A numerical algorithm for quasiconformal mappings (joint work with R. Michael Porter)

Hirokazu Shimauchi

Department of Humanities, Yamanashi Eiwa College

2015.11.8

Topology and Computer 2015, Nihon University, Tokyo

“日本一あたたかい大学”
山梨英和大学

Quasiconformal mappings in the plane

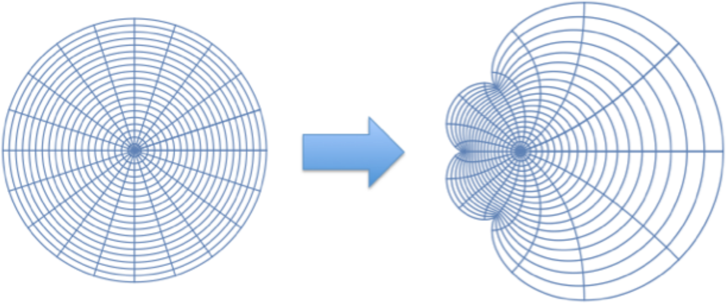
Conformal mappings

Let $\mathbb{C}$ be the complex plane and $\mathbb{D}$ the unit disk.
Recall:

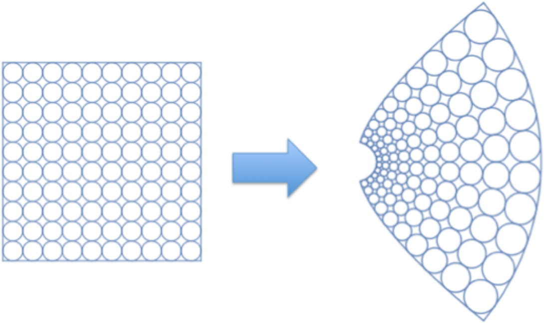
Theorem (Riemann mapping theorem)

Let $D \subsetneqq \mathbb{C}$ be a simply connected domain with $z_0 \in D$. Then there is a unique conformal map $f : D \to \mathbb{D}$ with $f(z_0) = 0$ and $f'(z_0) > 0$.

- Conformal mappings preserves local angles.



- Conformal mappings maps infinitesimal circles to infinitesimal circles.
  $f(z) - f(a) = f'(a)(z-a) + O((z-a)^2)$ in a neighborhood of $a \in D$.



Quasiconformal mapping

Definition

Let $K > 1$ and D, D' be the domains in the complex plane $\mathbb{C}$. An orientation-preserving homeomorphism $f : D \to D'$ is a K-**quasiconformal mapping** if f satisfies the following:

1. For any closed rectangle $R := \{z = x + iy \,|\, a \leq x \leq b, c \leq y \leq d\}$ in D, f is absolutely continuous on almost every horizontal and vertical line in R.
2. The dilatation condition

$$|f_{\overline{z}}(z)| \leq \frac{K-1}{K+1} |f_z(z)| \tag{1}$$

holds almost everywhere in D, where $f_z = (f_x - if_y)/2, f_{\overline{z}} = (f_x + if_y)/2$ and $z = x + iy$.

It follows from the definition that the quasiconformal mapping $f : D \to D'$ has partial derivatives $f_z, f_{\overline{z}}$ almost everywhere in D. Further f is differentiable a.e. in D, i.e. the real-linear approximation

$$f(z) - f(z_0) = f_z(z_0)(z - z_0) + f_{\overline{z}}(z_0)(\overline{z - z_0}) + o(|z - z_0|)$$

holds a.e. in $z_0 \in D$.

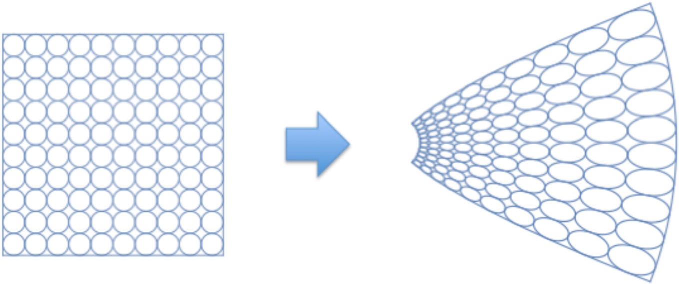


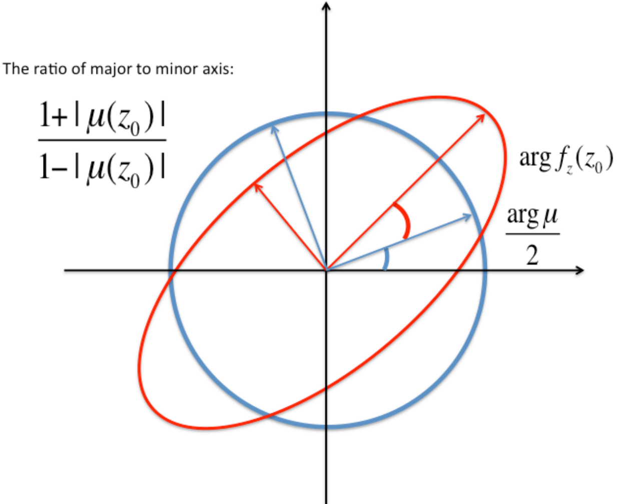
Figure. Quasiconformal mapping

- **The Beltrami coefficient** can be defined as

$$\mu(z) := \frac{f_{\overline{z}}(z)}{f_z(z)} \tag{2}$$

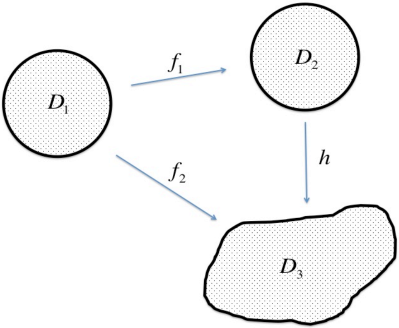
a.e. in D for a qausiconformal mapping f, which is a measure of non-conformality.

- If $\mu_f(z_0) = 0$ at $z_0 \in D$, f is conformal at z_0.



Proposition (Composition with conformal mapping)

Let D_1, D_2 be domains, and $\mu \in L^\infty(D_1)_1$. Assume that $f_1 : D_1 \to D_2$ is a μ-conformal mapping and $h : D_2 \to D_3$ a conformal mapping. Then $f_2 = h \circ f_1$ is μ-conformal.



Remark If we have self μ-conformal mappings of $\mathbb{D}$, then we can obtain μ-conformal mapping from $\mathbb{D}$ to arbitrary simply connected domains. Further many efficient methods for the numerical conformal mappings are known.

Set $L^\infty(D)_1 := \{\mu : D \to \mathbb{C} \mid \mu$ is measurable on D with $\|\mu\|_\infty < 1\}$.

Theorem (Measurable Riemann mapping theorem)

Let $\mu \in L^\infty(\mathbb{C})_1$. Then there exists a quasiconformal mappings $f : \mathbb{C} \to \mathbb{C}$ whose Beltrami coefficient coincides with μ almost everywhere in $\mathbb{C}$. This mapping is uniquely determined up to a conformal mapping of $\mathbb{C}$ onto itself.

Corollary

Let D, D' be bounded simply connected domains in $\mathbb{C}$ and $\mu \in L^\infty(D)_1$. Then there exists a quasiconformal mapping $f : D \to D'$ whose Beltrami coefficient coincides with μ almost everywhere in D. This mapping is uniquely determined up to a conformal mapping of D' onto it self.

- We say a quasiconformal mapping of D is μ-**comformal** ($\mu \in L^\infty(D)_1$) if its Beltrami coefficients coincide with μ almost everywhere in D.
- f is a homeomorphism from D to D' which satisfies **the Beltrami equation** $f_{\overline{z}} = \mu f_z$ on a.e. D.

The following lemma is useful for our setting.

Lemma (Good approximation lemma)

Let $\{\mu_n \in L^\infty(\mathbb{C})_1\}_{n\in\mathbb{N}}$ and satisfies $\|\mu_n\|_\infty \le k < 1$ for all $n \in \mathbb{N}$, and such that the pointwise limit $\mu(z) := \lim_{n\to\infty} \mu_n(z)$ exists almost everywhere. Let $f_n : \mathbb{C} \to \mathbb{C}$ be the μ_n-conformal mappings which fix $0, 1, \infty$. Then $f_n(z)$ converges to $f(z)$ uniformly, where f is the μ-conformal mapping which fix $0, 1, \infty$.

Basically this lemma states that: if the Beltarami coefficient of a quasiconformal mapping g approximate μ, then g approximate μ-conformal mapping.

Applications and known methods

Applications of quasiconformal mappings:

- Complex dynamics

  Gaidashev, D., and Yampolsky, M. (2007). Cylinder renormalization of Siegel disks. Experimental Mathematics, 16(2), 215-226.

- Medical image processing

  Lui, L. M., Wong, T. W., Zeng, W., Gu, X., Thompson, P. M., Chan, T. F., and Yau, S. T. (2012). Optimization of surface registrations using beltrami holomorphic flow. J. Scientific Computing, 50(3), 557-585.

- etc.

Setting

Quasiconformal self-mapping of the unit disk

Corollary

Let $\mathbb{D}$ be the unit disk and $\mu \in L^\infty(\mathbb{D})_1$. Then there exists unique μ-conformal mapping $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = f(1) - 1 = 0$.

For given $\mu \in L^\infty(\mathbb{D})_1$, we will construct an approximant of μ-conformal self-mapping of $\mathbb{D}$ with $f(0) = f(1) - 1 = 0$.

Triangulation of the unit disk in our setting

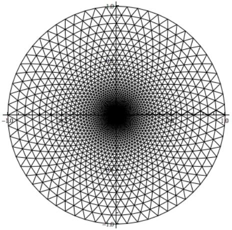


Figure. An triangulation of $\mathbb{D}$ which consists of 4096 2-simplices.

Definition (Triangulation of the unit disk)

We say a Euclidian simplicial complex T which consist of finite closed 2-simplices $\{\tau_i\}$ in $\mathbb{C}$ form **a triangulation of** $\mathbb{D}$ if:

1. $P := |T|$ is a closed simple jordan polygon whose vertices lies on the boundary of the unit disk $\partial\mathbb{D}$ where $|T|$ is the union of all 2-simplices in T,
2. each 1-face l_k of any 2-simplex τ_i of T is either:
   - an edge of P, or
   - there exists unique $j (j \neq i)$ such that l_k is an edge of a 2-simplex τ_j in T.

Observation

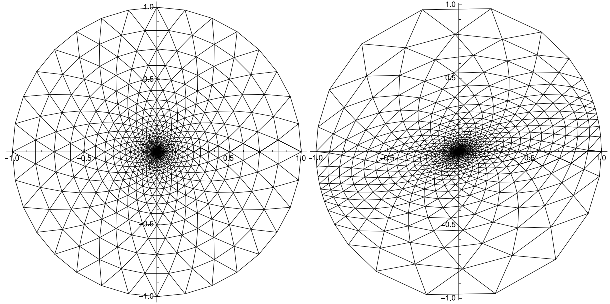


Figure: (Left) A triangulation T_z of the unit disk, (Right) A triangulation T_w of $\mathbb{D}$ which is simplicially equivalent to T_z.

- Let T_z, T_w be triangulations of $\mathbb{D}$.
- If T_z and T_w are simplicially equivalent, then the piecewise linear mapping $f : |T_z| \to |T_w|$ which sent 2-simplex in T_z to the corresponding 2-simplex in T_w linearly, is a homeomorphism between $|T_z|$ and $|T_w|$.
- We say f is **induced piecewise linear mapping** by T_z and T_w.

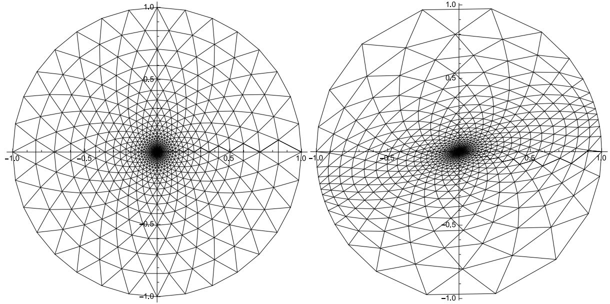


Figure: (Left) A triangulation T_z of the unit disk, (Right) a triangulation T_w of $\mathbb{D}$ which is simplicially equivalent to T_z.

- The Beltrami coefficients μ_f of $f : |T_z| \to |T_w|$ is defined on each interior of 2-simplex.
- Actually μ_f satisfies $\max_{\tau \in T_z} |\mu_f|_\tau - 0.3i| < 0.012$.
- f can be viewed as an approcimant of μ-conforaml mapping where $\mu(z) = 0.3\,i$.

Formulation of our problem

Definition

For given triangulation of the unit disk T_z, we say $f : |T_z| \to \mathbb{C}$ is in $PL(T_z)$ if f is continuous on $|T_z|$, and is linear on each 2-simplex in $|T_z|$.

We aim an algorithm as the following.

Input:

- $\mu \in L^\infty(\mathbb{D})_1$.
- A triangulations of the unit disk T_z whose vertices include 0 and 1.

Output:

- A triangulations of the unit disk $T_w \cong T_z$ whose vertices include 0 and 1 in suitable position, so that the Beltrami coefficient μ_g of the induced piecewise linear mapping $g : |T_z| \to |T_w| \in PL(T_z)$, reduce $\|\mu - \mu_g\|_\infty$ on each $\tau \in T_z$.

Algorithm

Logarithmic cordinates

- Let $\mu \in L^\infty(\mathbb{D})_1$.
- Set $\mu(z) := \frac{z^2}{\overline{z}^2}\overline{\mu(\frac{1}{\overline{z}})}$ for $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$.

Theorem (Recall: Measurable Riemann mapping theorem)

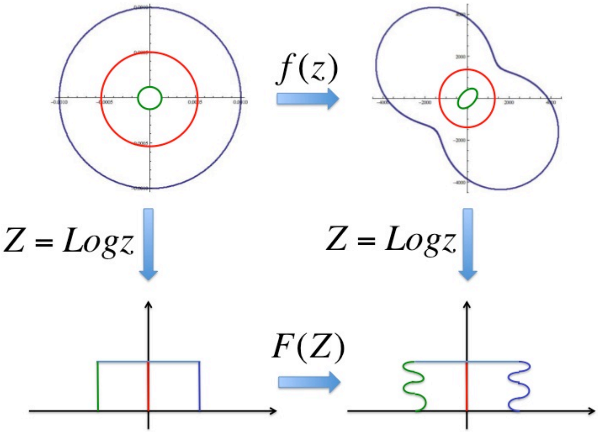
For given $\mu \in L^\infty(\mathbb{C})_1$, there exists unique μ-mappings $f : \mathbb{C} \to \mathbb{C}$ which fix $0, 1, \infty$.

We want to note that:

Corollary

If $\mu \in L^\infty(\mathbb{C})_1$ and $\overline{\mu(z)} = \mu(1/\overline{z})\overline{z}^2/z^2$, then the restriction of μ-conformal mapping $f^\mu : \mathbb{C} \to \mathbb{C}$ which fix 0 and 1 to the unit disk, is a self $\mu|_{\mathbb{D}}$-conformal mapping of $\mathbb{D}$ which fix 0 and 1.

- Actually $f|_{\mathbb{D}}$ is desired quasiconformal mapping where $f : \mathbb{C} \to \mathbb{C}$ is μ-conformal mapping.



- Take the logarithmic coordinates $Z = \log z$. Then $F(Z) := \log f(e^Z)$ have the symmetry with respect to the imaginary axis.
- First we approximate $F(Z) := \log f^{\mu}(e^Z)$ on a finite rectangle.

Take $M, N \in \mathbb{N}$. We define $(M+1)N$ vertices

$$Z_{j,k} = \frac{\sqrt{3}\,\pi\, j}{N} + \frac{2\pi(k + (j \bmod 2)/2)}{N} i \tag{3}$$

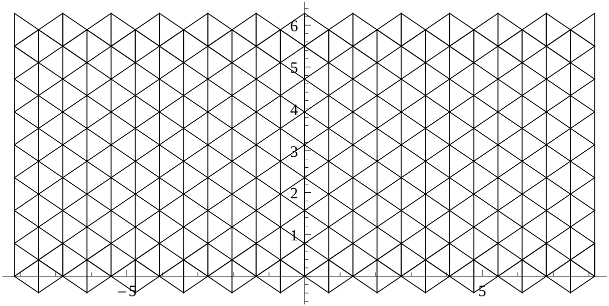
for $-M \leq j \leq 0$ and $0 \leq k \leq N-1$. Our mesh contains $M \times N$ *rightward pointing* 2*-simplexes* defined by

$$\tau^{+}_{j,k} = \begin{cases} \text{Conv}(Z_{j-1,k-1},\ Z_{j-1,k},\ Z_{j,k}), & j \text{ even}, \\ \text{Conv}(Z_{j-1,k},\ Z_{j-1,k+1},\ Z_{j,k}), & j \text{ odd}, \end{cases} \tag{4}$$

for $-M+1 \leq j \leq 0$ where Conv(Z_1, Z_2, Z_3) is the 2-simplex which vertices are Z_1, Z_2, Z_3. There are also $M \times N$ *leftward pointing* 2*-simplexes*

$$\tau^{-}_{j,k} = \begin{cases} \text{Conv}(Z_{j+1,k-1},\ Z_{j+1,k},\ Z_{j,k}), & j \text{ even}, \\ \text{Conv}(Z_{j+1,k},\ Z_{j+1,k+1},\ Z_{j,k}), & j \text{ odd}, \end{cases} \tag{5}$$

for $-M \leq j \leq -1$.



In the case the triangles $\tau_{jk}^{\pm}$ are equilateral. We extend this mesh symmetrically to the right half-plane as

$$Z_{j,k} = \varrho(Z_{-j,k})$$

where ϱ is the reflection of the imaginary axis

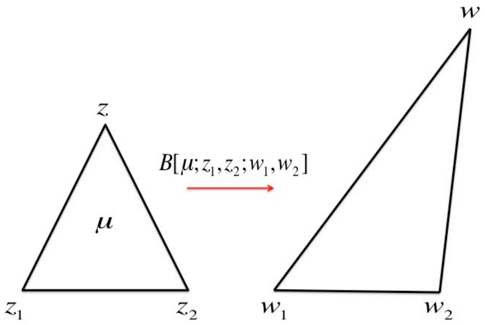
$$\varrho(Z) = -\overline{Z}. \tag{6}$$

Now we have $(2M+1)N$ vertices and $4MN$ 2-simplexes. We say this **the basic mesh** in the logarithmic coordinates.

Observation: linear quasiconformal mapping

Proposition

Let $z_1, z_2, w_1, w_2 \in \mathbb{C}$ *with* $z_1 \neq z_2$ *and* $w_1 \neq w_2$*. For given complex constant* $\mu \in \mathbb{D}$*, there is a unique* μ*-conformal affine linear mapping* $B(z) = B[\mu; z_1, z_2; w_1, w_2](z)$ *which sends* z_i *to* w_i $(i = 1, 2)$*.*



Let μ, a, b be complex constants with $a \neq 0$, $|\mu| < 1$. We consider the μ-conformal real-linear mapping

$$L_\mu(z) \quad := \quad \frac{z + \mu\bar{z}}{1+\mu}. \tag{7}$$

Proposition

$B(z)$ is given by

$$\begin{aligned} B(z) &= w_1 + \frac{w_2 - w_1}{L_\mu(z_2 - z_1)} L_\mu(z - z_1) \\ &= \frac{L_\mu(z_2 - z)}{L_\mu(z_2 - z_1)} w_1 + \frac{L_\mu(z_1 - z)}{L_\mu(z_1 - z_2)} w_2. \end{aligned}$$

Remark We note that the coefficients of w_1, w_2 in the last expression are never 0, 1, or ∞ if z_1, z_2, z_3 are distinct.

Corollary

Let μ-conformal affine linear map takes z_1, z_2, z_3 to w_1, w_2, w_3 respectively. Then the following holds:

$$L_\mu(z_2 - z_3)\, w_1 + L_\mu(z_3 - z_1)\, w_2 + L_\mu(z_1 - z_2)\, w_3 = 0. \tag{8}$$

Corollary

Let $z_i \in \mathbb{C}$ *(*$i = 1, 2, 3$*) noncollinear and* $w_i \in \mathbb{C}$ *(*$i = 1, 2, 3$*) noncollinear. There is a unique affine linear mapping which sends* z_i *to* w_i $(i = 1, 2, 3)$*. Further its Beltrami coefficient is equal to*

$$\mu = -\frac{(z_2 - z_1)(w_3 - w_1) - (z_3 - z_1)(w_2 - w_1)}{(\overline{z_2} - \overline{z_1})(w_3 - w_1) - (\overline{z_3} - \overline{z_1})(w_2 - w_1)}. \tag{9}$$

Beltrami coefficient of F

The Beltrami coefficients $\nu(Z)$ of $F(Z)$ are given as follows by the chain rule for quasiconformal mappings :

$$\nu(Z) = \mu(e^Z)\frac{e^{\overline{Z}}}{e^Z} = \mu(e^Z)e^{-2i\,\mathrm{Im}\,Z}, \mathrm{Re}\ Z < 0. \tag{10}$$

Using ν, we set the Beltrami coefficients as $\nu(Z) = \overline{\nu(\varrho(Z))}$ for Re $Z > 0$.

We will write $\nu^{\pm}_{j,k}$ for the average value of $\nu(Z)$ on the 2-simplexes $\tau^{\pm}_{j,k}$. It is useful for numerical work to take the average of $\nu(Z)$ over the three vertices as an approximation of this average, at least when ν is continuous. Let us note that

$$\nu_{jk} = \overline{\nu_{-j,k}}, \quad j > 0. \tag{11}$$

Triangle equations

For all rightward pointing 2-simplicies $\tau_{jk}^{+} \in T_{M,N} := \{\tau_{j,k}^{\pm}\}$, we construct following MN linear equations:

$$a_{jk}^{+} W_{jk} + b_{jk}^{+} W_{j-1,k} + c_{jk}^{+} W_{j-1,k+1} = 0 \tag{12}$$

where

$$\begin{aligned}
a_{jk}^{+} &= \begin{cases} L_{\nu_{jk}}(Z_{j-1,k-1} - Z_{j-1,k}), & j \text{ even}, \\ L_{\nu_{jk}}(Z_{j-1,k} - Z_{j-1,k+1}), & j \text{ odd}, \end{cases} \\
b_{jk}^{+} &= \begin{cases} L_{\nu_{jk}}(Z_{j-1,k} - Z_{j,k}), & j \text{ even}, \\ L_{\nu_{jk}}(Z_{j-1,k+1} - Z_{jk}), & j \text{ odd}, \end{cases} \\
c_{jk}^{+} &= \begin{cases} L_{\nu_{jk}}(Z_{j,k} - Z_{j-1,k-1}), & j \text{ even}, \\ L_{\nu_{jk}}(Z_{jk} - Z_{j-1,k}), & j \text{ odd}. \end{cases}
\end{aligned} \tag{13}$$

Further MN linear equations for the leftward pointing 2-simplexes τ^-_{jk} are constructed Corollary 4,

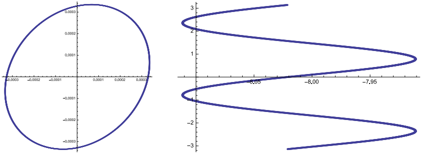
$$a^-_{jk} W_{jk} + b^-_{jk} W_{j+1,k-1} + c^-_{jk} W_{j+1,k} = 0 \quad (14)$$

where

$$\begin{aligned} a^-_{jk} &= \begin{cases} L_{\nu_{jk}}(Z_{j+1,k-1} - Z_{j+1,k}), & j \text{ even}, \\ L_{\nu_{jk}}(Z_{j+1,k} - Z_{j+1,k+1}), & j \text{ odd}, \end{cases} \\ b^-_{jk} &= \begin{cases} L_{\nu_{jk}}(Z_{j+1,k} - Z_{j,k}), & j \text{ even}, \\ L_{\nu_{jk}}(Z_{j+1,k+1} - Z_{j,k}), & j \text{ odd}, \end{cases} \\ c^-_{jk} &= \begin{cases} L_{\nu_{jk}}(Z_{j,k} - Z_{j+1,k-1}). & j \text{ even}, \\ L_{\nu_{jk}}(Z_{j,k} - Z_{j+1,k}), & j \text{ odd}. \end{cases} \end{aligned} \quad (15)$$

Remark We have totally $4MN$ triangle equations.

Boundary equations



Originally, the image of the infinitesimal circle by a quasiconformal mapping, is infinitesimal ellipse. The shape of this ellipse is depend on the Beltrami coefficients. Under this situation, we will add the following equations.

Let d_k be the images of these points under the real-linear mapping L_{μ_0}, i.e.

$$d_k = L_{\mu_0}(e^{Z_{-M,k}}) = r_{-M} L_{\mu_0}(e^{2\pi i k/N}), \quad 0 \leq k \leq N-1,$$

where μ_0 denotes the average value of $\mu(z)$ inside this circle. These vertices lie on an ellipse. We want a condition that the image of $\mathcal{C}$ is unknown complex nonzero constant multiple of the ellipse $\{d_k\}$. Hence the boundary equations which achieve above condition are the following $2(N-1)$ equations

$$\begin{aligned} W_{-M,k} - W_{-M,k-1} &= D_k, \\ W_{M,k} - W_{M,k-1} &= \overline{D_k}, \end{aligned} \qquad (16)$$

where $D_k = \log \mathcal{C} d_k - \log \mathcal{C} d_{k-1} = \log d_k - \log d_{k-1}$ and $1 \leq k \leq N-1$. The magnitude of r_{-M} does not influence the value of D_k.

Nomalization

Finally, for normalization of the solution we add one more equation,

$$W_{0,0} = 0, \tag{17}$$

which is self-symmetric. This says that $F(0) = 0$, or equivalently, $f(1) = 1$.

Associated linear system

In argument above, we construct $n_{\mathrm{e}} = 4MN + 2(N-1) + 1$ complex linear equations for the $n_{\mathrm{v}} = (2M+1)N$ unknown variables W_{jk}, $-M \le j \le M$, $0 \le k \le N-1$. Let $p = p(j,k)$ be an fixed bijection from the set of index pairs $\{(j,k)\}$ to the range $1 \le p \le n_{\mathrm{v}}$. Using this bijection p, we will rename the variables in a single vector $\boldsymbol{W}$ with

$$\boldsymbol{W} := \{W_p\} = \{W_{j,k}\} \tag{18}$$

for the convenience. The linear system now takes the form:

$$\boldsymbol{A}\boldsymbol{W} = \boldsymbol{B} \tag{19}$$

where $\boldsymbol{A} = (A_{j,k})$ is the $n_{\mathrm{e}} \times n_{\mathrm{v}}$-type complex matrix and $\boldsymbol{B} = (B_k)$ is a complex vector of length n_{e}. When we take a pair of N, M, the mesh $\{Z_{jk}\}$ is fixed, and linear system above is defined. We will say that this linear system $(\boldsymbol{A}, \boldsymbol{B})$ is the *associated linear system* to the collection of ν-values $\{\nu_{jk}\}$. The coefficients depend both on ν_{jk} and Z_{jk}.

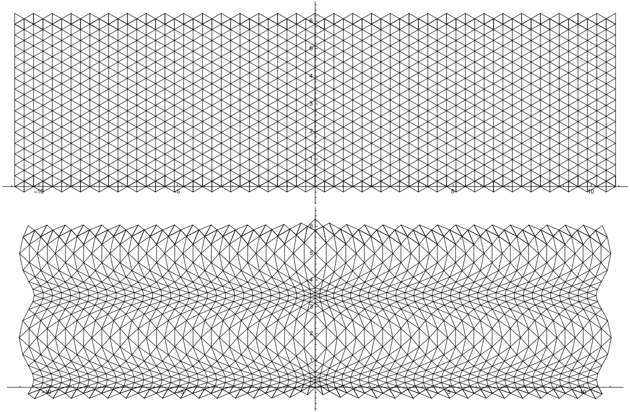
Since our linear system is over determined, we chose the standard least squares method for the approximation.

Definition (Least squares)

Let $m, n \in \mathbb{N}$ with $m > n$. Let $\boldsymbol{AW} = \boldsymbol{B}$ an overdetermined linear system where $\boldsymbol{A} \in \boldsymbol{M}_{m,n}(\mathbb{C})$, $B \in \mathbb{C}^m$ and unknown vector $\boldsymbol{W} \in \mathbb{C}^n$. We call $\boldsymbol{W}$ is the least squares solution of $(\boldsymbol{A}, \boldsymbol{B})$ if $\boldsymbol{W}$ minimize the residual vector $\|\boldsymbol{AW} - \boldsymbol{B}\|_2$.

Lemma

The least squares solution $\boldsymbol{W} = \{W_{j,k}\}$ *(*$-M \leq j \leq M$*,* $0 \leq k \leq N-1$*) of the associated linear system* $(\boldsymbol{A}, \boldsymbol{B})$ *exists uniquely. Furthermore* $\boldsymbol{W}$ *satisfies the following symmetric relation:* $\boldsymbol{A}\overleftrightarrow{\boldsymbol{W}} = \boldsymbol{A}\varrho(\boldsymbol{W})$ *where* $\overleftrightarrow{W}_{j,k} = W_{-j,k}$*,* $\varrho(\boldsymbol{W}) = \{\varrho(W_{j,k})\}$ *and* ρ *is defined by s*(6)*, i.e. the entries of* $\boldsymbol{W}$ *satisfy the symmetry* $W_{-j,k} = \rho(W_{j,k})$*. In particular, the values* $\{W_{0,k}\}$ *are purely imaginary.*



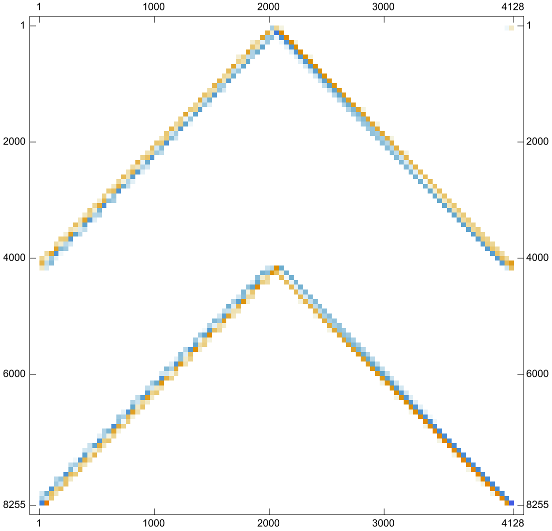


Figure. An exmaple of $\boldsymbol{A}$ $(M = 64, N = 32, \mu(z) = 0.3)$.

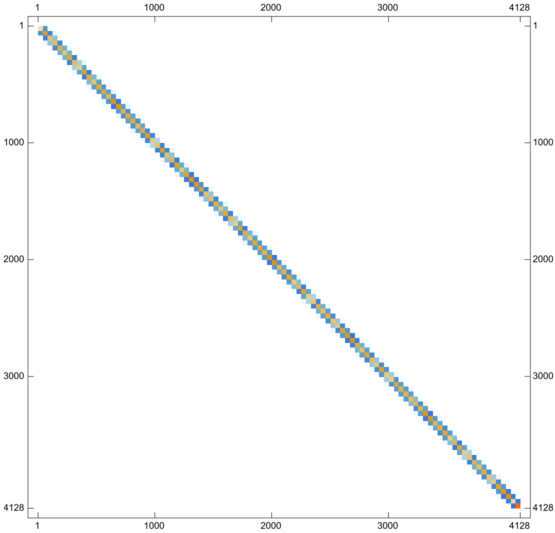
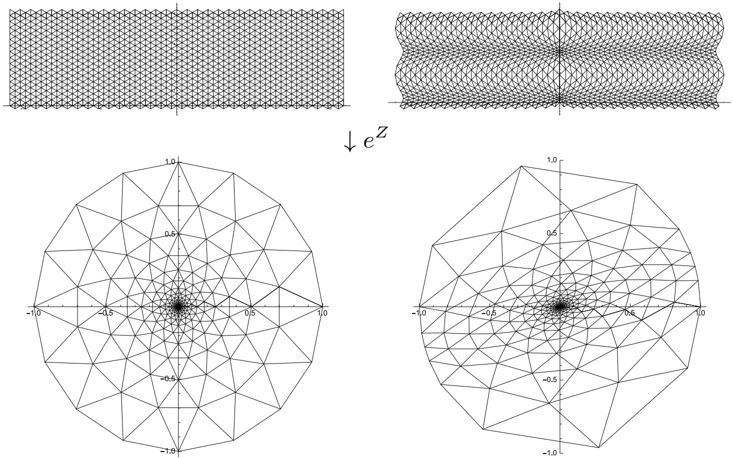


Figure. ${}^tA \cdot A$ $(M = 64, N = 32, \mu(z) = 0.3)$.



Finally, we apply the exponential mapping to the vertices of $\{Z_{j,k}\}$ and $\{W_{j,k}\}$, and then we take the piecewise linear mapping which is induced by the corresponding between the two simplices.

The algorithm is summarized as follows.

Algorithm

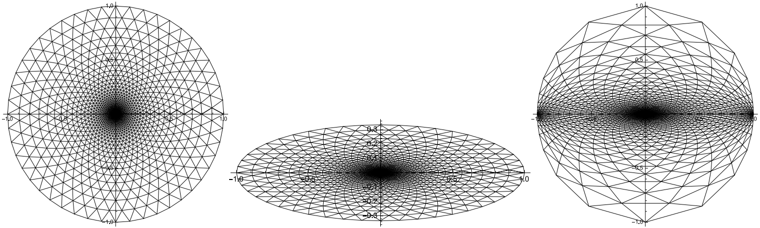
Input: The Beltrami coefficient $\mu \in L^\infty(\mathbb{D})_1$ and the dimensions M, N for a simplicial complex $\{Z_{j,k}\}$ in the Z-plane.

1. Calculate the averages of the Beltrami coefficients $\nu_{j,k}$ on each triangle in the logarithmic coodrinates.
2. Calculate the coefficients of the associated linear system $(\boldsymbol{A}, \boldsymbol{B})$ of $\{\nu_{jk}\}$ and $T_{M,N}$ as prescribed .
3. Calculate the least squares solution $\boldsymbol{W}$ to the associated linear system $(\boldsymbol{A}, \boldsymbol{B})$, and arrange the entries of $\boldsymbol{W}$ to form the mesh $\{W_{jk}\}$.
4. 4. Calculate $w_{jk} = \exp W_{jk}$ for $-M \leq j \leq 0$ and $0 \leq k \leq N-1$.

Output: The piecewise linear mapping such that $z_{jk} \mapsto w_{jk}$ where $z_{jk} = \exp Z_{jk}$.

Numerical experiments

Constant Beltrami coefficients



The image of the circle $|z| = 1$ under the mapping L_μ is an ellipse with semiaxes 1, $(1 - |\mu|)/(1 + |\mu|)$ slanted in the directions $(1/2)\arg\mu$, $(1/2)(\arg\mu + \pi)$ respectively. This ellipse is sent by the conformal linear mapping $H_{1/(2\sqrt{\mu}),0}$ to the ellipse with semiaxes a, b. Then the ellipse is transformed conformally to the unit disk, by an explicit formula for the conformal mapping to $\mathbb{D}$ from this ellipse.

The algorithm was applied for the constant Beltrami derivatives $\mu = 0.1,\ 0.3,\ 0.5,\ 0.7$, and meshes defined by $N = 16$, 32, 48, 64, 72, 84, with M equal to the least multiple of 4 no less than $N \log N/(\pi\sqrt{3})$.

(M, N)	(12,16)	(24,32)	(36,48)	(52,64)	(60,72)	(72,84)
$\mu =$0.1	0.012	0.0031	0.0014	0.0008	0.0006	0.0004
$\mu =$0.3	0.0274	0.007	0.0031	0.0018	0.0014	0.001
$\mu =$0.5	0.0615	0.0205	0.0109	0.0065	0.0051	0.0038
$\mu =$0.7	0.2439	0.1201	0.0856	0.0627	0.053	0.0412

Table: The maximum of the absolute errors between the solutions and the real values of some constant Beltrami derivative and $M \approx N \log N/(\pi\sqrt{3})$.

In the last case there are 24359 equations in 14196 variables. It took 4 seconds to solve the full set of equations.

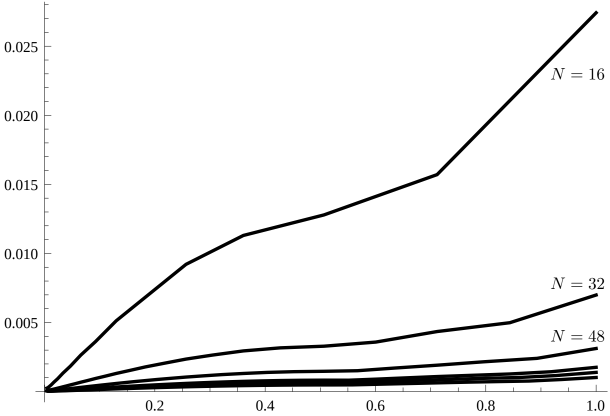


Figure: Numerical errors of algorithm for different values of (M, N) with $\mu = 0.3$.

The horizontal axis indicates the distance $r_j = |z_{jk}|$ of the z-points from the origin; the vertical axis gives the maximum discrepancy (over k) of the calculated value of w_{jk} from the true value.

Radial quasiconformal mappings

Let $\varphi\colon [0,1] \to [0,1]$ be an increasing diffeomorphism of the unit interval. Then the radially symmetric function

$$f(z) = \varphi(|z|)e^{i \arg z} = \varphi(|z|)\frac{z}{|z|} \tag{20}$$

has Beltrami derivative equal to

$$\mu(z) = \frac{|z|\varphi'(z)/\varphi(z) - 1}{|z|\varphi'(z)/\varphi(z) + 1}\frac{z}{\overline{z}} \tag{21}$$

when $z \neq 0$. As an illustration we will take

$$\varphi(r) = (1 - \cos 3r)/(1 - \cos 3).$$

The resulting Beltrami derivative satisfies $\|\mu\|_\infty = 0.65$ approximately.

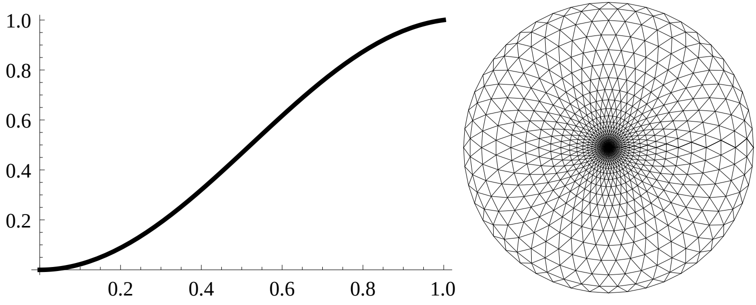


Figure. φ and T_w

(M,N)	$(12, 16)$	$(24, 32)$	$(36, 48)$	$(52, 64)$	$(60, 72)$	$(72, 84)$
Error	0.0398	0.0135	0.0058	0.0034	0.0027	0.0020

Table. The domain points z_{jk} on the real axis were selected, and the values of w_{jk} produced by the algorithm were compared with with the true values $\varphi(|z_{jk}|)$.

Sectrial quasiconformal mappings

In a similar spirit, we let $\psi\colon [0,2\pi] \to [0,2\pi]$ be an increasing diffeomorphism. Write $\widetilde{\psi}(e^{i\theta}) = e^{i\psi(\theta)}$. Then the sectorially symmetric function

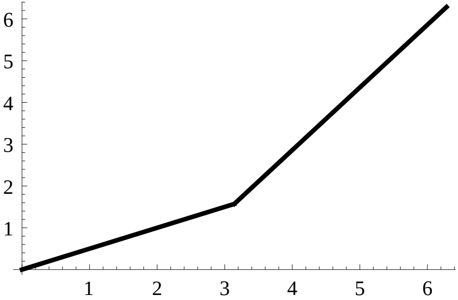
$$f(z) = |z|\,\widetilde{\psi}\left(\frac{z}{|z|}\right) \tag{22}$$

has Beltrami derivative equal to

$$\mu(z) = \frac{1-\psi'(\theta)}{1+\psi'(\theta)}\,\frac{z}{\bar{z}} \tag{23}$$

when $z \neq 0$. As an example we will take

$$\psi(\theta) = \begin{cases} \frac{\theta}{2}, & 0 \le \theta \le \pi, \\ \frac{\pi}{2} + \frac{3(\theta-\pi)}{2}, & \pi \le \theta \le 2\pi. \end{cases}$$



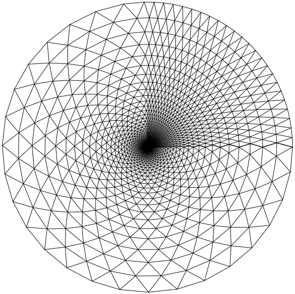


Figure. $\psi(\theta)$ and T_w

(M,N)	$(12, 16)$	$(24, 32)$	$(36, 48)$	$(52, 64)$	$(60, 72)$	$(72, 84)$
Error	0.0712	0.0362	0.0251	0.0193	0.0173	0.0150

Table. The arguments of the final boundary values on the unit circle were compared with the true values $\psi(\theta)$.

Large dilatation and oscilation

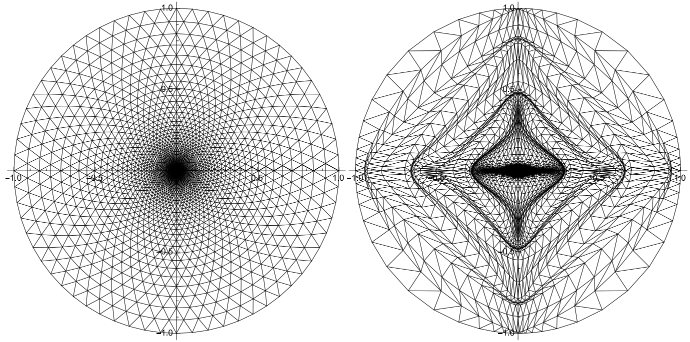


Figure. $\mu(z) = 0.9\sin(20|z|)$.

Trivial Beltrami coefficient

Let $\mu \in L^\infty(\mathbb{D})$ with $\|\mu\|_\infty < 1$. If the corresponding normalized solution f^μ satisfies $f^\mu(z) = z$ on the unit circle, μ called a trivial Beltrami coefficient. Trivial Beltrami coefficients play an important role in the theory of Teichmüller space. T. Sugawa showed a criterion for the triviality of the Beltrami coefficients, and gave an example for a trivial Beltrami coefficient. Let N be a non-negative integer and $a_j(t)$ $(1 \leq j \leq N)$ be essentially bounded measurable functions in $t \geq 0$ so that

$$\mu(z) := \sum_{j=0}^{N} a_j\left(-\log|z|\right)\left(\frac{z}{|z|}\right)^{j+2}$$

satisfies $\|\mu\|_\infty < 1$. Then his results implies that μ is a trivial Beltrami coefficient. For the experiment, we chose

$$a_j(z) := \frac{2}{3}\left(\frac{\sin 10z}{2}\right)^{j+1}.$$

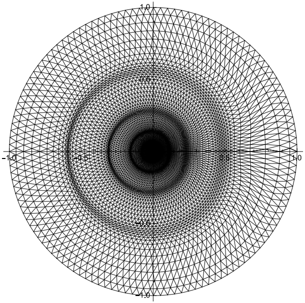


Figure: The result made by our algorithm with trivial coefficient μ_1.

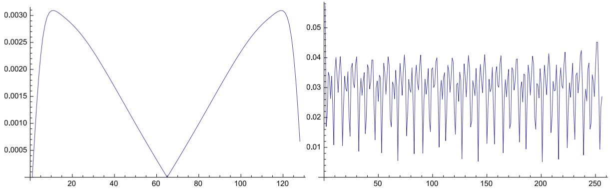


Figure: The errors of the boundary values (left), the difference between the induced Beltrami coefficients to μ_1 (right).

Convergence

Theorem (Porter, S ,2014)

Let $s \in \mathbb{N}$ *and* $M_s, N_s \in \mathbb{N}$ *be strictly increasing sequences which satisfy*

$$c_1 N_s \log N_s \leq M_s \leq c_2 N_s \log N_s \tag{24}$$

for constants c_1, c_2 *where* $c_1 > 1/(\pi\sqrt{3})$. *If* $\mu \in L_\infty(\mathbb{D})_1 \cup C^1(\mathbb{D})$, *then the following holds.*

i. *If* s *is large enough, the points* $\{z_{j,k}^{(s)}\}$ *and the points* $\{w_{j,k}^{(s)}\}$ *produced by the algorithm form the vertex sets of triangulations* $T_z^{(s)}$ *and* $T_w^{(s)}$ *of the unit disk* $\mathbb{D}$. *Furthermore, for any fixed compact set* $K \subset \text{int}\, \mathbb{D}$, $K \subset |T_z^{(s)}|$ *and* $K \subset |T_w^{(s)}|$ *hold when* s *is large enough.*

ii. *The mappings* $f^{(s)}$ *converge to the* μ*-conformal mapping* f *normalized by* $f(0) = f(1) - 1 = 0$ *uniformly on compact subsets of* $\mathbb{D}$ *as* $s \to \infty$.

Input:

- $\mu \in C^1(\mathbb{D}) \cap L^\infty(\mathbb{D})_1$.
- $M_s, N_s \to \infty$ as $s \to \infty$ with $c_1 N_s \log N_s \leq M_s \leq c_2 N_s \log N_s$.

Output:

- $\{g^{(s)} \in PL(T_z^{(s)})\}$ s.t. $g^{(s)} \to f^\mu$ as $s \to \infty$.

Remark We conjecture that the condition $\mu \in C^1$ is overly restrictive by the numerical experiments.

Key point of the proof

- Calculations show that:

$$\|\boldsymbol{A}_s \boldsymbol{W}_s - \boldsymbol{B}_s\|_2 \to 0.$$

- We obtain F_s is a local homeomorphism and the image of the boundary form a Jordan polygon.
- Using the good approximation lemma and the following lemma, we obtain the convergence.

Lemma

Let $T_z := \{\tau_j\}$ be a triangulation of the unit disk $\mathbb{D}$ with $P_z := |T_z|$ is a simple jordan polygon of k sides. Suppose $f\colon |T_z| \to \mathbb{D} \in PL(T_z)$ preserve the orientation on each $\tau \in T_z$ and maps $\partial|T_z|$ to a boundary of a simple polygon P_w with k sides on the unit circle homeomorphically. Then the secant map induced by f and T_z, is a orientation preserving homeomorphism from $|T_z|$ to P.

Modified algorithm

Theorem (Principal Solution)

Let $\mu : \mathbb{C} \to \mathbb{C}$ be a measurable function with compact support and $\|\mu\|_\infty < 1$. Set $\mu(z) = 0$ on the outside of its support. Then there exists unique μ-conformal mapping $f : \mathbb{C} \to \mathbb{C}$ of the plane which satisfies $f(0) = 0$ and $f(z) = z + O(1/z)$ as $z \to \infty$.

We modify our algorithm for the principal solution to the Beltrami equation with compactly supported μ.

Let $X_m = \{z \in \mathbb{C} : -m < \text{Re } z < m, -m < \text{Im } z < m\}$ be a square. X_m subdivide into $(2n)^2$-squares (same size, edge length = m/n).

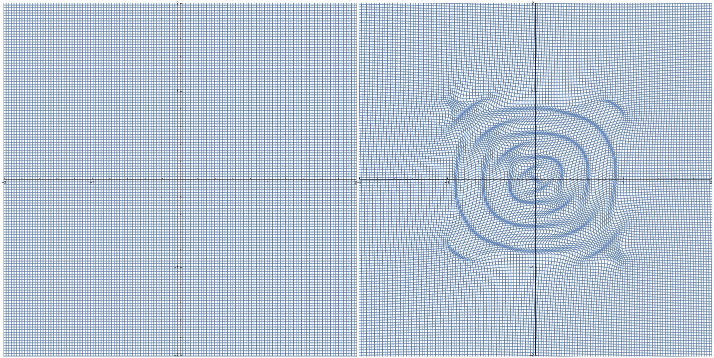


Figure: Approximation of Principal solution

We use the triangle equation with the condition $f(z) \approx z$ as $z \to \infty$ and normalization $f(0) = 0$.

Conclusion

Conclusion

- We propose an algorithm for numerical quasiconformal mappings.
- The approximant converge to the true solution at least in the case where the Beltrami coefficients are in C^1.
- For the details:
  Porter, R. Michael, and Hirokazu Shimauchi. *Numerical solution of the Beltrami equation via a purely linear system*, submitted.

Thak you very much for your attention !