Jørgensen numbers of some Kleinian groups on the boundary of the Schottky space

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Contents

Motivation

- A Kleinian group is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$.
- $\mathrm{Isom}^+(\mathbb{H}^3) \cong PSL(2, \mathbb{C})$ (by the Poincaré extention)
- Σ : a complete hyperbolic 3-manifold.

$$\rho : \pi_1(\Sigma) \longrightarrow PSL(2, \mathbb{C})$$

  ρ is faithful $\Longrightarrow \rho(\pi_1(\Sigma))$ is a torsion-free Kleinian group.
- Conversely, if G is a torsion-free Kleinian group, $\mathbb{H}^3/G$ is a complete hyperbolic 3-manifold.

We want to classify Kleinian groups in order to classify hyperbolic manifolds.

Elementary groups

Definition

A group $G < PSL(2, \mathbb{C})$ is elementary.
$\overset{def}{\Longleftrightarrow}$ There is a finite G-orbit in $\overline{\mathbb{H}^3}$.

Lemma

$G < PSL(2, \mathbb{C})$: a Kleinian group.

- G is elementary.
  $\Leftrightarrow$ The limit set $\Lambda(G)$ consists of 0, 1, or 2 points.
- G is non-elementary.
  $\Leftrightarrow$ The limit set $\Lambda(G)$ is an infinite set.

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Classification theorem of elementary Kleinian groups

Theorem

A torsion-free elementary Kleinian group is conjugate to one of the following:

(1) A parabolic cyclic group: $\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$

(2) A parabolic abelian group rank 2: $\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \right\rangle$ $(\Im\alpha > 0)$

(3) A loxodromic cyclic group: $\left\langle \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\rangle$ $(|\lambda| \neq 0, 1)$

Jørgensen's theorems

- When is a non-elementary group discrete (i.e, a Kleinian group)?

Theorem (Jørgensen)

$G < PSL(2, \mathbb{C})$: non-elementary group.
G is a Kleinian. $\Leftrightarrow \forall f, g \in G$, $\langle f, g \rangle$ is a Kleinian.

Theorem (Jørgensen)

$G := \langle f, g \rangle < PSL(2, \mathbb{C})$ is a non-elementary Kleinian group.
Then,

$$J(f, g) := |tr^2(f) - 4| + |tr(fgf^{-1}g^{-1}) - 2| \geq 1$$

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Jørgensen number

Definition

$G < PSL(2, \mathbb{C})$: a non-elementary 2-generator group.

$$J(G) := \inf\{J(f, g) \mid G = \langle f, g \rangle\}$$

is called the *Jørgensen number of* G.

- Now we consider the following problem :

Problem

r : a real number with $r \geq 1$.
When is there a non-elementary Kleinian group whose Jørgensen number is equal to r?

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Oichi-Sato's theorems

- Oichi and Sato show that :

Theorem (Oichi-Sato)

r : a positive integer.
Then, there is a non-elementary Kleinian group G $s.t.$ $J(G) = r$.

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r : a real number with $r > 4$.
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Problems

From these theorems, we have the following problems:

Problem (Oichi-Sato)

$r \in (1, 4)$: a non-integer.
When is there a non-elementary Kleinian group whose Jørgensen number is equal to r?

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r : a real number with $r > 4$.
Is there a non-elementary Kleinian group other than the already known groups whose Jørgensen number is equal to r?

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The Riley slice

- We will consider the one-parameter family of non-elementary Kleinian groups generated by two parabolic transformations $X,\ Y_\rho$.
- We normalize so that $\mathrm{Fix}\,(X) = \{0\}, \mathrm{Fix}\,(Y_\rho) = \{\infty\}$:

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ Y = \begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix} (\rho \in \mathbb{C}).$$

- $G_\rho := \langle X,\ Y_\rho \rangle$.

Definition

The *Riley slice* is defined by

$$\mathcal{R} := \{\rho \in \mathbb{C} \mid G_\rho \text{ is free and } \Omega\,(G_\rho)\,/G_\rho \text{ is 4-times punctured sphere}\}.$$

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- From L.Keen, C.Series, The Riley slice of the Schottky spaces

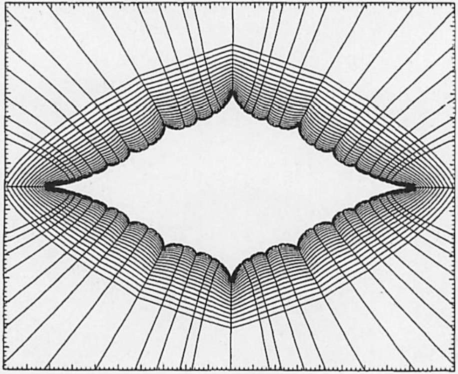

FIG. 1. The Riley slice

A characterization of Riley slice

Theorem (Maskit-Swarup)

A Kleinian group of 2nd kind generated by two parabolic transformations is geometrically finite.

Theorem (Maskit)

G : a geometrically finite two-generator free Kleinian group with a parabolic transformation.
Then, G is a "point" of the boundary of Schottky space of rank 2.

Corollary

Every $\rho \in \mathcal{R}$, G_ρ is a point of the boundary of Schottky space of rank 2.

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Jørgensen number on the Riley slice

Proposition

$\forall \rho \in \mathcal{R},\ J(G_\rho) = |\rho|^2$

For the Riley slice, Keen and Series show that :

Theorem (Keen-Series)

If $\rho_0 \in \mathcal{R},\ r > |\rho_0|$, then there is $\rho \in \mathcal{R}$ s.t. $|\rho| = r$.

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An extensition of Oichi-Sato's theorem

We obtain the following theorem :

Theorem (Y)

$\forall r \geq \frac{5}{2}$, $\exists \rho \in \mathcal{R}$ s.t. $J(G_\rho) = r$.
In particular,
$\forall r \geq \frac{5}{2}$, there is a group on the boundary of the Schottky space rank 2 s.t. $J(G) = r$.

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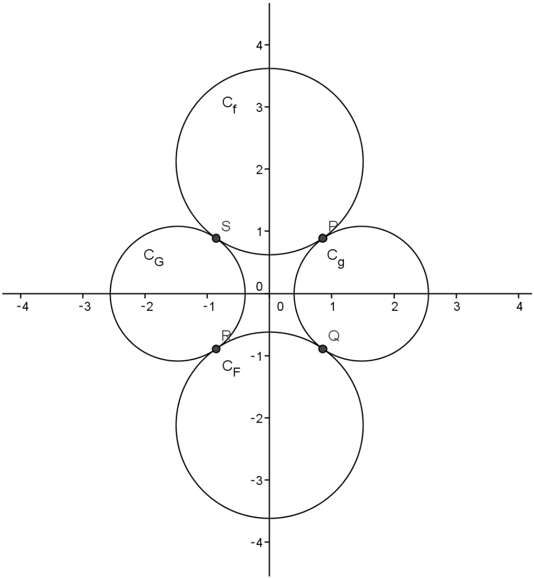
Kissing Schottky group

Definition

$f, \ g$: loxodromic transformations,
$\{C_f^+, C_f^-\}, \ \{C_g^+, C_g^-\}$: circle pairings corresponding to $f, \ g$.
$G = \langle f, \ g \rangle$ is a *kissing Schottky group*.
$\overset{def}{\Longleftrightarrow}$

- G is a free group.
- $C_f^+ \cap C_g^+ = \{P\}$, $C_g^+ \cap C_f^- = \{Q\}$, $C_f^- \cap C_g^- = \{R\}$, $C_g^- \cap C_f^+ = \{S\}$ (one points).


4
3
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0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3
4
Cf
CG
Cg
CF
S
P
R
Q

Once punctured torus group

Lemma

A kissing Schottky group is a “point” on the boundary of the Schottky space.

Theorem (cf. [Indra's pearls])

$G = \langle f, g \rangle$: K.S.group.
G is a once punctured torus group.

$$\iff f(Q) = P,\ f(R) = S,\ g(R) = Q,\ g(S) = P.$$

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Jørgensen numbers of once punctured torus groups

Corollary (Y)

$G = \langle f, g \rangle$: once punctured torus group.
Then $J(G) > 4$.

Future problems

Problem

$r > 4$: a real number.
When is there a K.S. once punctured torus group whose Jørgensen number is equal to r?

Problem

What is happened when Jørgensen number is equal to 4?

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Thank you for your attention!