Discreteness of some 4-generator Möbius groups

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Background

- **Möbius transformation**

A Möbius transformation is a rational function of the form

$$T(z) = \frac{az+b}{cz+d} \ (a, b, c, d \in \mathbb{C}, ad - bc = 1)$$

With the Möbius transformation T, we can associate

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

A Möbius transformation is a composition of

・translation ・inversion ・homothety ・rotation

- **Character variety**

$F_2 = \langle X, Y\rangle$: free group of rank 2

$$\mathrm{SL}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\}$$

$Hom(F_2, SL(2,\mathbb{C})) = \{f : F_2 \to SL(2,\mathbb{C}) | f \ is \ homomorphism.\}$

$f \sim g \Leftrightarrow {}^{\exists}M \in SL(2,\mathbb{C}),$
$Mf(X)M^{-1} = g(X), Mf(Y)M^{-1} = g(Y)$

$\chi = Hom(F_2, SL(2,\mathbb{C}))/\sim$: **Character variety**

χ can be identified with $\mathbb{C}^3$.

$[f] \in \chi \longrightarrow (tr\ f(X), tr\ f(Y), tr\ f(XY))$

remark $\mathrm{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{a+d}$

- **Discrete**

A subgroup G of SL$(2,\mathbb{C})$ is called discrete

if ${}^{\exists}$*a neighborhood* N *of* $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ *such that* $G \cap N = \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}$

- **Maskit slice**

$(x, y, z) \in \chi$

$f_{x,y,z} \in Hom(F_2, SL(2, \mathbb{C}))$

$$tr\ f_{x,y,z}(X) = x,\ tr\ f_{x,y,z}(Y) = y,\ tr\ f_{x,y,z}(XY) = z$$

$\{(x, y, z) \in \chi \mid x = 2, x^2 + y^2 + z^2 = xyz, f_{x,y,z}\ \textit{is injective and}\ f_{x,y,z}(F_2)\ \textit{is discrete.}\}$

$= \{(2, y, y + 2i) \mid f_{2,y,y+2i}\ \textit{is injective and}\ f_{2,y,y+2i}(F_2)\ \textit{is discrete.}\}$: **Maskit slice**

- **Farey triangulation**

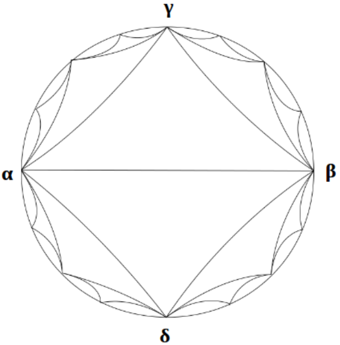
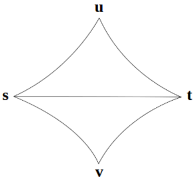


Figure : a triangulation of unit disk

< Definition >
V:the set of vertices of Farey triangulation

A map ρ:V$\to\mathbb{C}$ is called Markov map
if $\rho(v)=\rho(s)\cdot\rho(t)-\rho(u)$,
where stu and stv are triangles of Farey triangulation.



We fix a triangle $\alpha\beta\gamma$.
For $[f]\in\chi$, the Markov map associated with [f] is given by
$(\rho(\alpha),\rho(\beta),\rho(\gamma))=(tr\ f(X), tr\ f(Y), tr\ f(XY))$.

Purpose

⟨Maskit slice⟩

- If $x^2 + y^2 + z^2 = xyz$,
  $^{\exists}$algorithm which decides discreteness of $f_{x,y,z}$.
- In $\{(x, y, z) \mid x = 2, x^2 + y^2 + z^2 = xyz\}$, the Maskit slice is connected.
- In the Maskit slice, there are pleating rays.

⟨4-generator Möbius groups⟩

- Plumbing construction

(A construction of 4-generator Möbius groups from $f_{x,y,z}$ in Maskit slice)

- Parametrized by$(x, y, z) = (2, y, y + 2i)$ *and* $t \in \mathbb{C}$ (plumbing parameter)

Q What is the shape of discrete subset in $\{(y, t) | y, t \in \mathbb{C}\}$?

Calculation

- **4-generator Möbius groups**

$f_{x,y,z} \in Maskit$

$g_{y,t} \;:\; F_4 \;\rightarrow\; SL(2,\mathbb{C})$ is defined as follows.

$(x,y,z) = (2,y,y+2i)$, $F_4 = \langle X,Y,Z,W \rangle$: free group of rank 4

$$g_{y,t}(X) = \begin{pmatrix} x - \frac{y}{z} & \frac{x}{z^2} \\ x & \frac{y}{z} \end{pmatrix}, \; g_{y,t}(Y) = \begin{pmatrix} y - \frac{x}{z} & \frac{-y}{x^2} \\ -y & \frac{x}{z} \end{pmatrix}$$

$$g_{y,t}(Z) = Pg_{y,t}(X)P^{-1}, g_{y,t}(W) = Pg_{y,t}(Y)P^{-1}$$

$$P = \begin{pmatrix} i & -ti \\ 0 & i \end{pmatrix}$$

$g_{y,t}$ is obtained from $f_{x,y,z}$ by plumbing construction.

- **Parameter y**

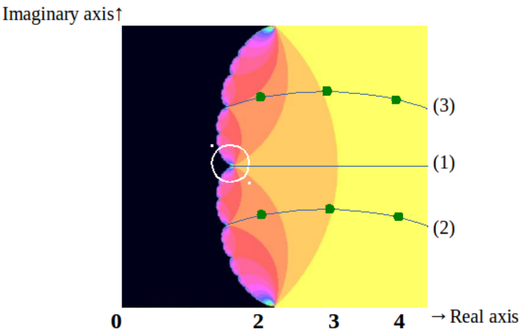


Figure 1

Figure : Maskit slice y-plane

- colored black : non discrete
- otherwise : discrete and injective

- y=2,3,4
- y=2+i,3+i,4+i,$\sqrt{3}$+i (1)
- y=0.58i+1.69,0.59i+1.82,0.60i+1.93 (2)
- y=1.41i+1.69,1.40i+1.82,1.39i+1.93 (3)

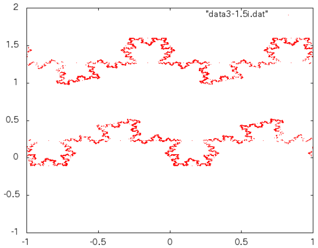
- **Deciding the discreteness of Möbius groups**

⟨1⟩ limit set

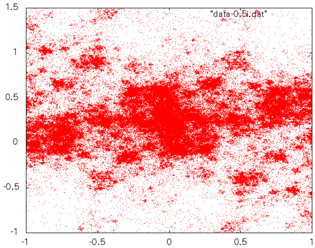
G : a subgroup of SL(2,$\mathbb{C}$)

- The limit set of G : plot all fixed points of length at most N.(N=7)

$\langle example(x = 2, y = 3, z = 3 + 2i)\rangle$



discrete(t=1.5i)



non discrete(t=0.5i)

$\langle 2 \rangle$ Jorgensen's inequality

If $A, B \in SL(2, \mathbb{C})$ generate a non-elementary discrete group,then

$$|tr(A)^2 - 4| + |tr(ABA^{-1}B^{-1}) - 2| \geq 1$$

In our computer experiment, we fix

$$A = g_{y,t}(X)g_{y,t}(Y)g_{y,t}(X)^{-1}g_{y,t}(Y)^{-1}$$

$\langle 3 \rangle$ elliptic element

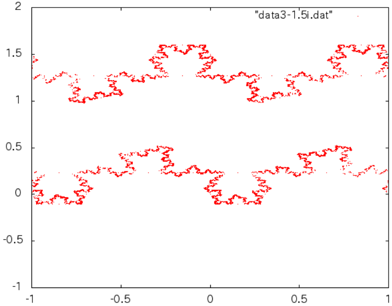
・ $A \subset SL(2, \mathbb{C})$ is elliptic if trA is real and $-2 < trA < 2$.

Let A be an elliptic element.

・ If $trA = 2\cos\frac{\pi}{n}$ for some $n \in \{2, 3, 4 \cdots\}$,then $\langle A \rangle$ is discrete.
Otherwise $\langle A \rangle$ is not discrete.

⟨4⟩ Drawing the real loci

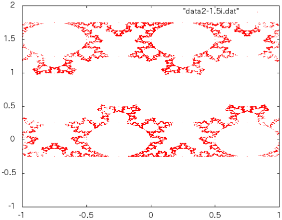
- Draw the limit set of 4-generator Möbius groups $(g_{y,t})$
- Find $s \in F_4$,
  whose fixed point $Fix(g_{y,t}(s))$ is "highest" in the "lower limit set".
- $tr(t) = tr\ g_{y,t}(s)$: polynomial on t (We fix y and s)
- Draw curves $tr(t) \in \mathbb{R}$
- s : "highest words"



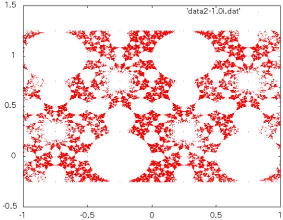
discrete(t=1.5i)

Results

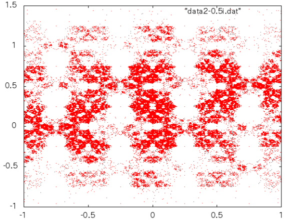
⟨1⟩limit set (x=2,y=2,z=2+2i)



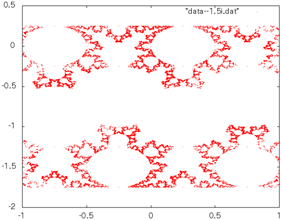
t=1.5i



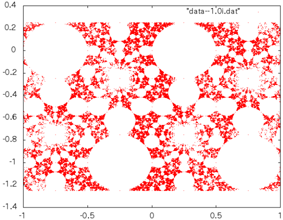
t=1.0i



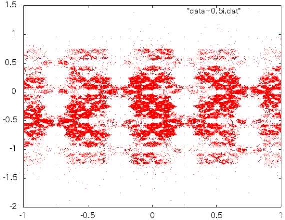
t=0.5i



t=-1.5i



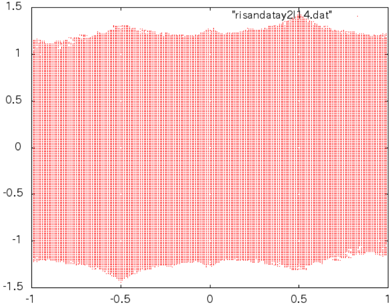
t=-1.0i



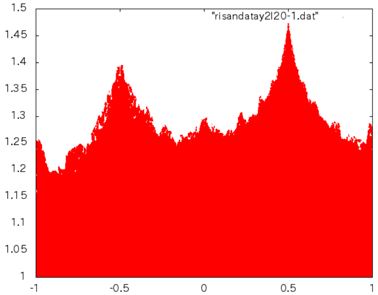
t=-0.5i

The boundary is in the vicinity of +1.0i,-1.0i.

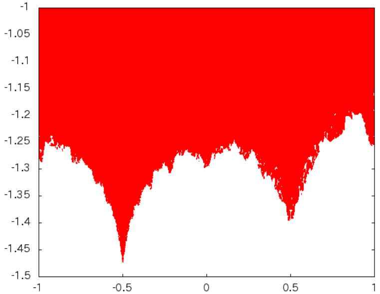
⟨2⟩Jorgensen (x=2,y=2,z=2+2i)



Jorgensen



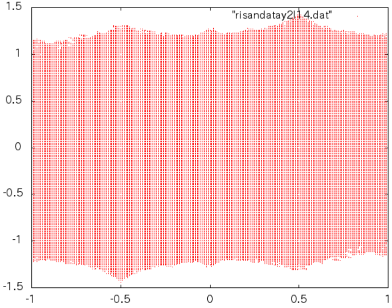
t-imag(plus)



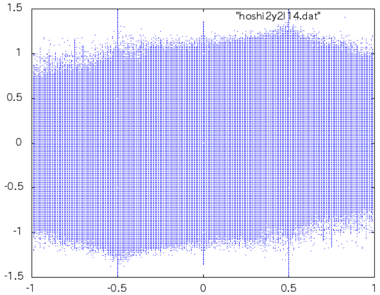
t-imag(minus)

Figure of Jorgensen's inequality,
up and down point symmetry
(t-imag(plus), t-imag(minus))

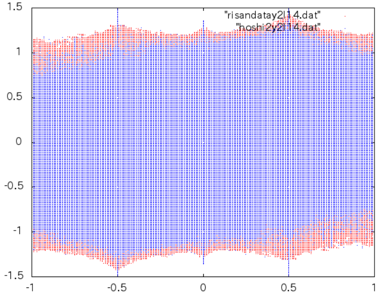
⟨3⟩elliptic (x=2,y=2,z=2+2i)



Jorgensen



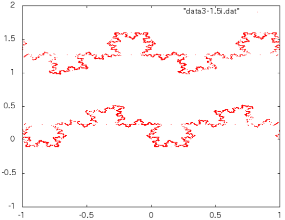
elliptic



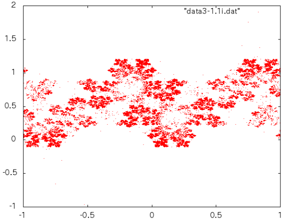
Jorgensen + elliptic

Comparison of
the figure of Jorgensen's inequality
and the figure of elliptic element
: roughly match

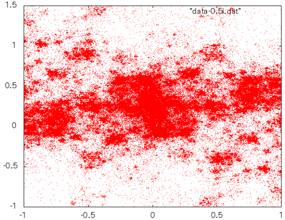
- limit set (x=2,y=3,z=3+2i)



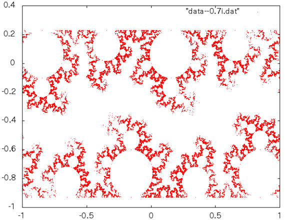
t=1.5i



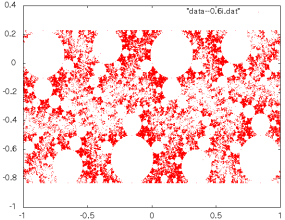
t=1.1i



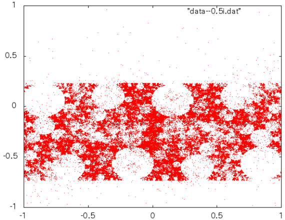
t=0.5i



t=-0.7i



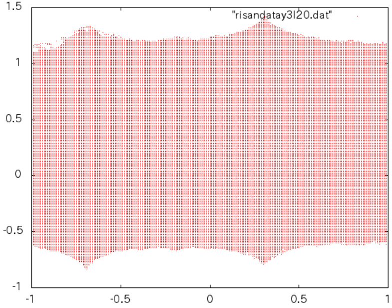
t=-0.6i



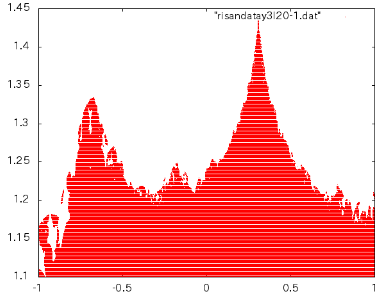
t=-0.5i

The boundary is in the vicinity of +1.1i,-0.6i.

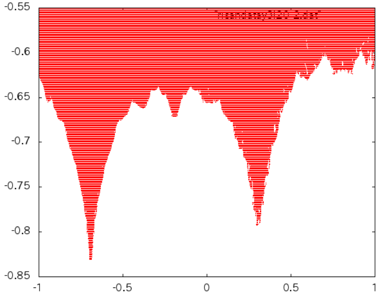
- Jorgensen (x=2,y=3,z=3+2i)



Jorgensen



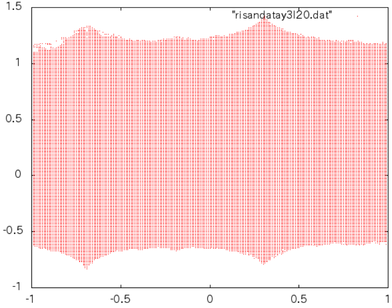
t-imag(plus)



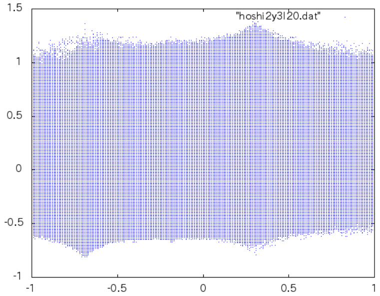
t-imag(minus)

Figure of Jorgensen's inequality (t-imag(plus), t-imag(minus))

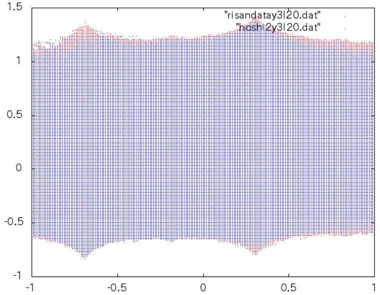
- elliptic (x=2,y=3,z=3+2i)



Jorgensen



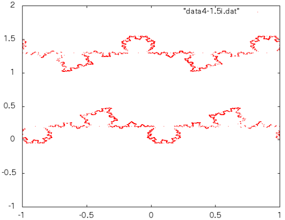
elliptic



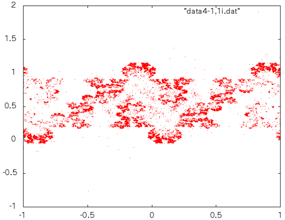
Jorgensen + elliptic

Comparison of
the figure of Jorgensen's inequality
and the figure of elliptic element
: roughly match

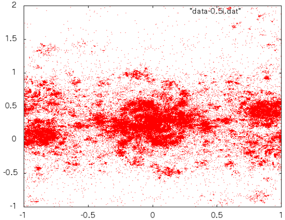
- limit set (x=2,y=4,z=4+2i)



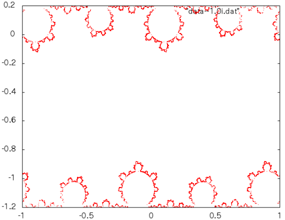
t=1.5i



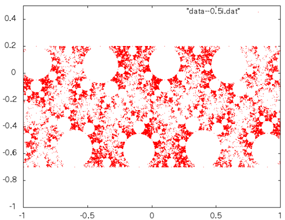
t=1.1i



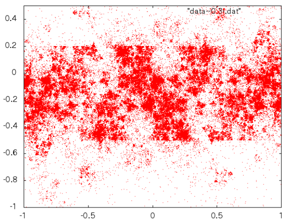
t=0.5i



t=-1.0i



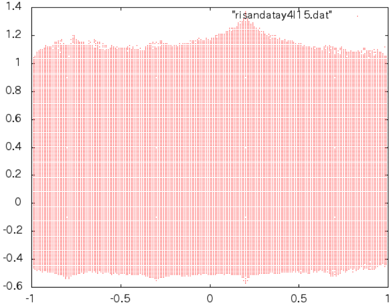
t=-0.5i



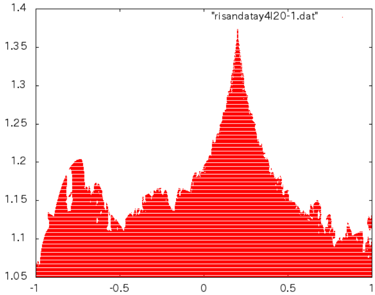
t=-0.3i

The boundary is in the vicinity of +1.1i,-0.5i.

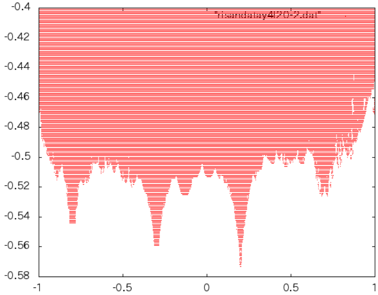
- Jorgensen (x=2,y=4,z=4+2i)



Jorgensen



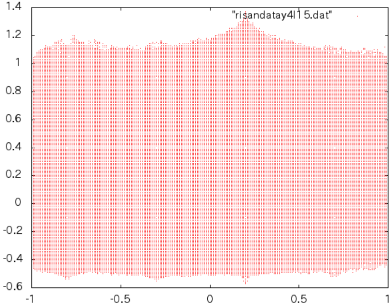
t-imag(plus)



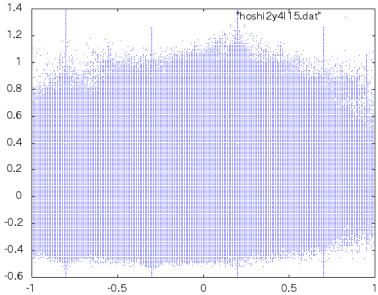
t-imag(minus)

Figure of Jorgensen's inequality,
different up and down
(t-imag(plus), t-imag(minus))

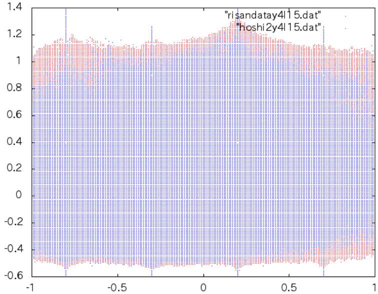
- elliptic (x=2,y=4,z=4+2i)



Jorgensen



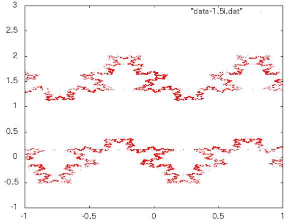
elliptic



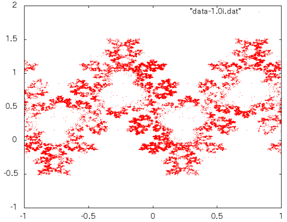
Jorgensen + elliptic

Comparison of
the figure of Jorgensen's inequality
and the figure of elliptic element
: roughly match

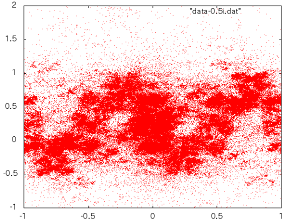
- limit set $(x=2, y=2+i, z=2+3i)$



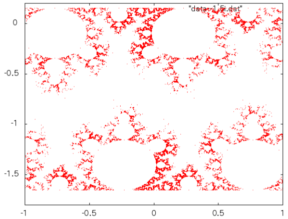
t=1.5i



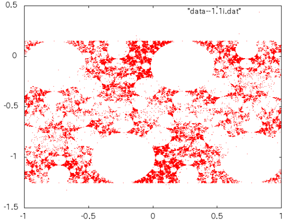
t=1.0i



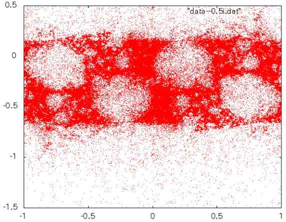
t=0.5i



t=-1.5i



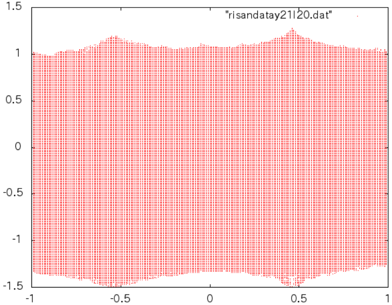
t=-1.1i



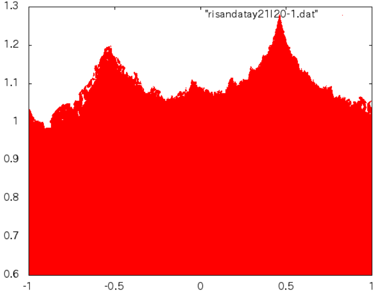
t=-0.5i

The boundary is in the vicinity of +1.0i,-1.1i.

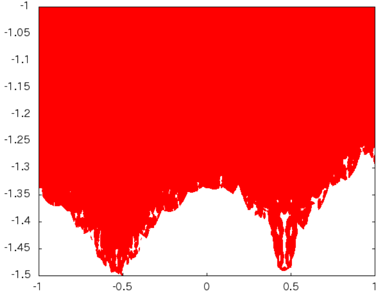
- Jorgensen $(x=2, y=2+i, z=2+3i)$



jorgensen



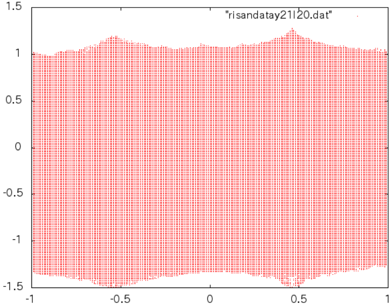
t-imag(plus)



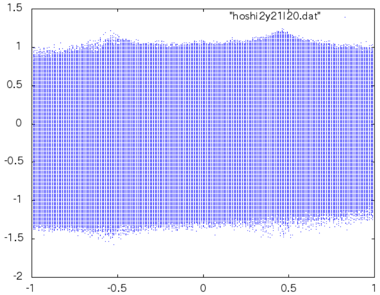
t-imag(minus)

Figure of Jorgensen's inequality,
different up and down
(t-imag(plus), t-imag(minus))

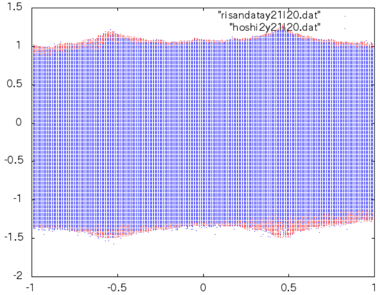
• elliptic (x=2,y=2+i,z=2+3i)



Jorgensen



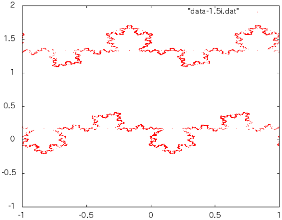
elliptic



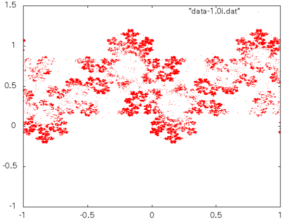
Jorgensen + elliptic

Comparison of
the figure of Jorgensen's inequality
and the figure of elliptic element
: roughly match

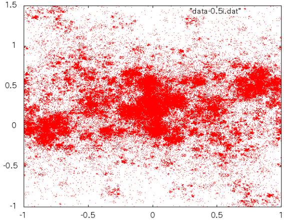
- limit set (x=2,y=3+i,z=3+3i)



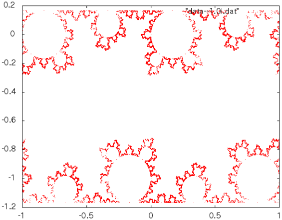
t=1.5i



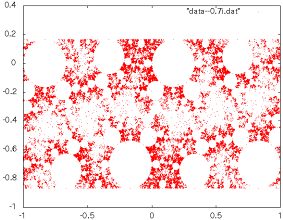
t=1.0i



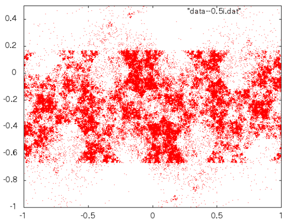
t=0.5i



t=-1.0i



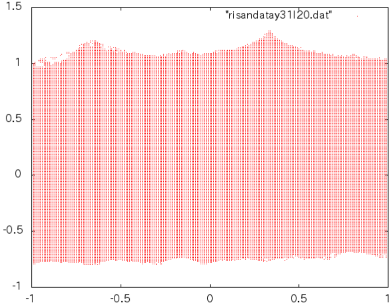
t=-0.7i



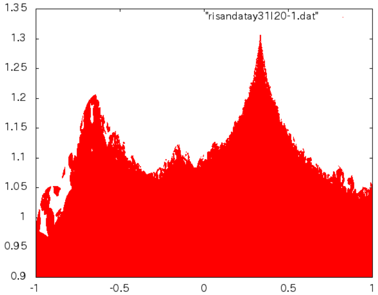
t=-0.5i

The boundary is in the vicinity of +1.0i,-0.7i.

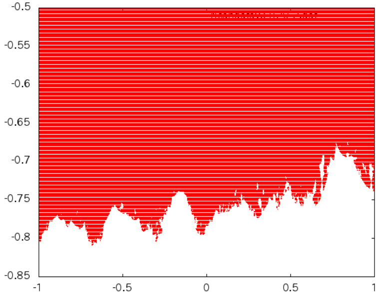
- Jorgensen $(x=2, y=3+i, z=3+3i)$



Jorgensen



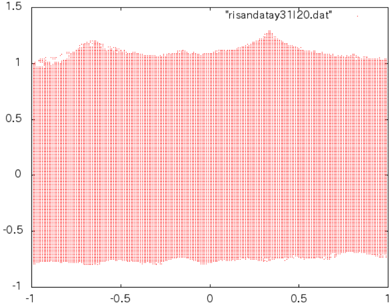
t-imag(plus)



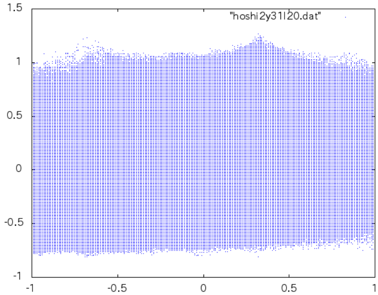
t-imag(minus)

Figure of Jorgensen's inequality, different up and down (t-imag(plus), t-imag(minus))

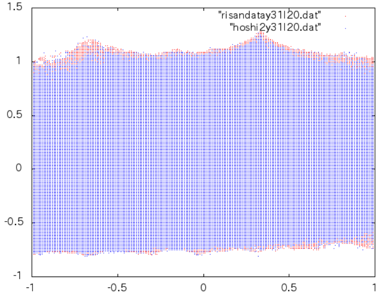
- elliptic $(x=2, y=3+i, z=3+3i)$



Jorgensen



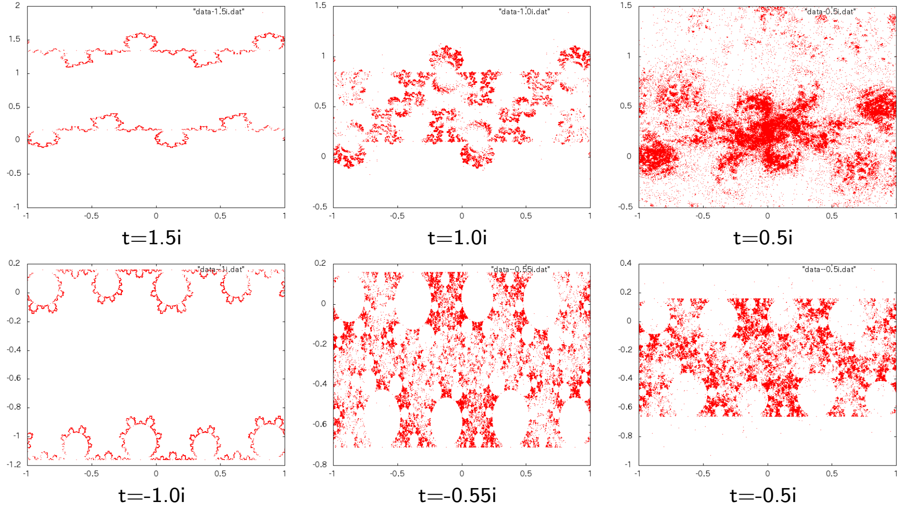
elliptic



Jorgensen + elliptic

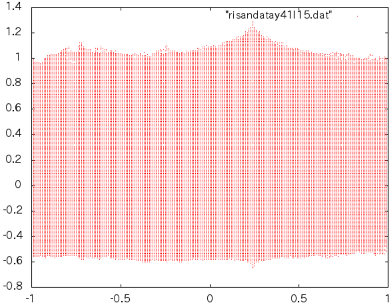
Comparison of
the figure of Jorgensen's inequality
and the figure of elliptic element
: roughly match

• limit set (x=2,y=4+i,z=4+3i)

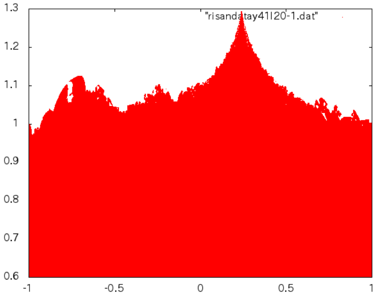


The boundary is in the vicinity of 1.0i,-0.55i.

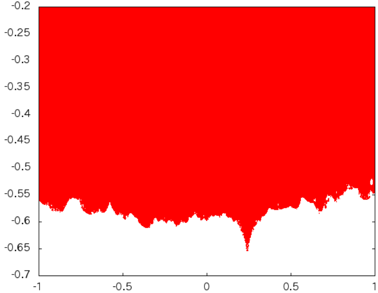
- Jorgensen $(x=2, y=4+i, z=4+3i)$



jorgensen



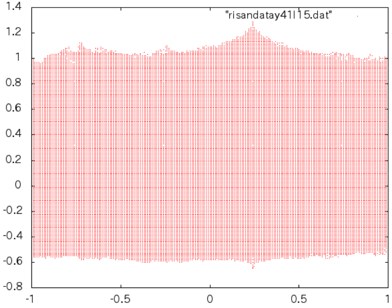
t-imag(plus)



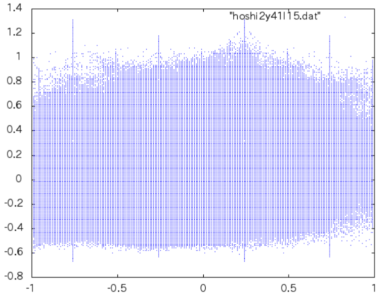
t-imag(minus)

Figure of Jorgensen's inequality,
different up and down
(t-imag(plus), t-imag(minus))

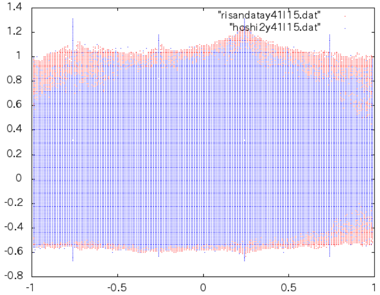
- elliptic (x=2,y=4+i,z=3+3i)



Jorgensen



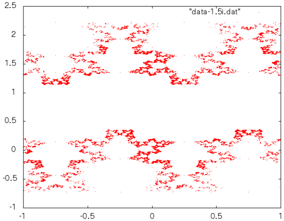
elliptic



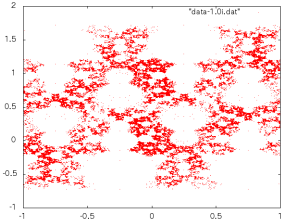
Jorgensen + elliptic

Comparison of
the figure of Jorgensen's inequality
and the figure of elliptic element
: roughly match

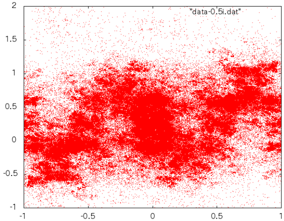
- limit set $(x=2, y=\sqrt{3}+i, z=\sqrt{3}+3i)$



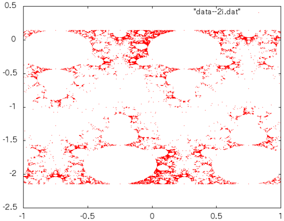
t=1.5i



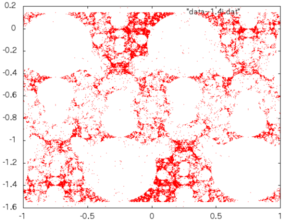
t=1.0i



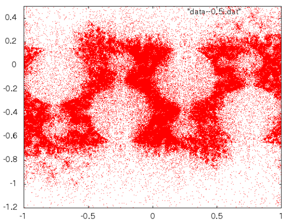
t=0.5i



t=-2.0i



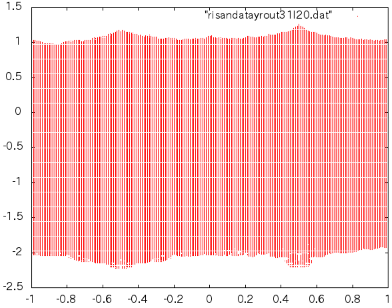
t=-1.4i



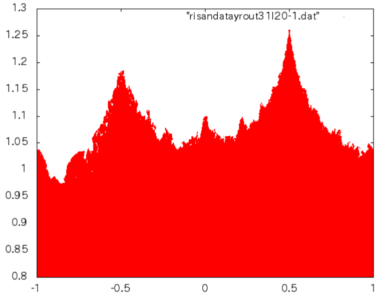
t=-0.5i

The boundary is in the vicinity of +1.0i,-1.4i.

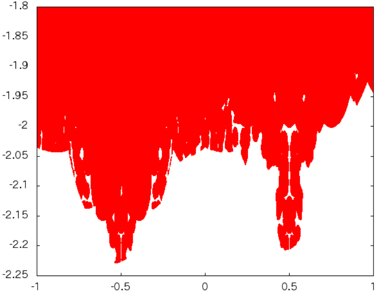
- Jorgensen $(x=2, y=\sqrt{3}+i, z=\sqrt{3}+3i)$



Jorgensen



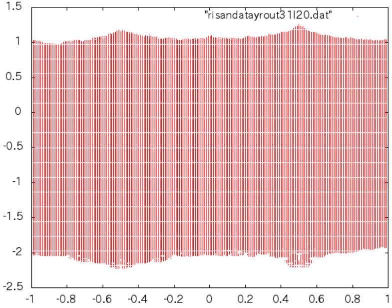
t-imag(plus)



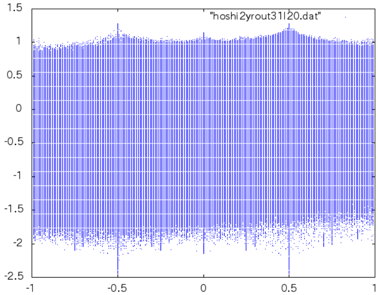
t-imag(minus)

Figure of Jorgensen's inequality,
different up and down
(t-imag(plus), t-imag(minus))

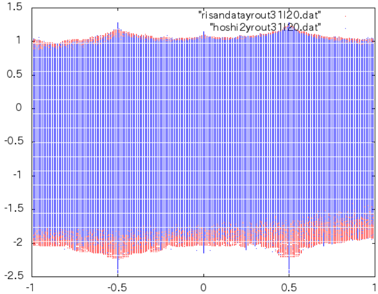
- elliptic $(x=2, y=\sqrt{3}+i, z=\sqrt{3}+3i)$



Jorgensen



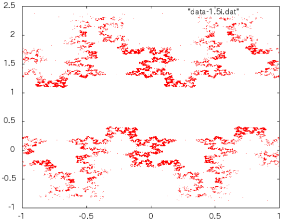
elliptic



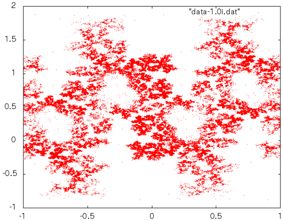
Jorgensen + elliptic

Comparison of
the figure of Jorgensen's inequality
and the figure of elliptic element
: roughly match

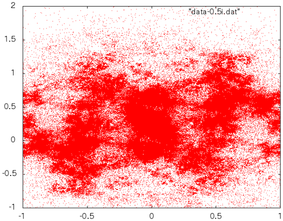
- limit set ($x=2, y=0.58i+1.69, z=y+2i$)



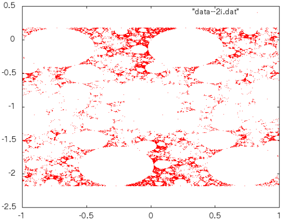
$t=1.5i$



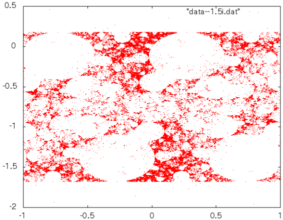
$t=1.0i$



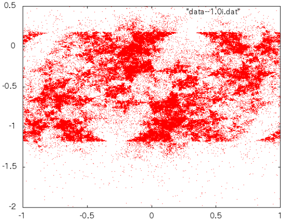
$t=0.5i$



$t=-2.0i$



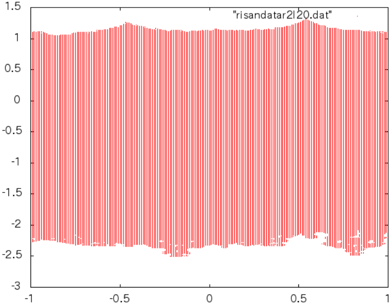
$t=-1.5i$



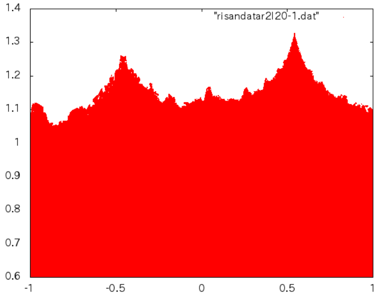
$t=-1.0i$

The boundary is in the vicinity of $+1.0i, -1.5i$.

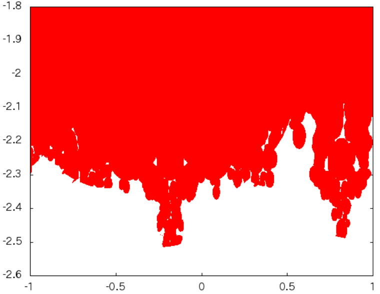
- Jorgensen (x=2,y=0.58i+1.69)



Jorgensen



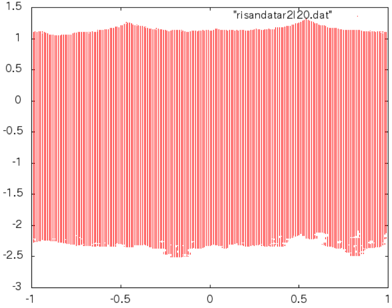
t-imag(plus)



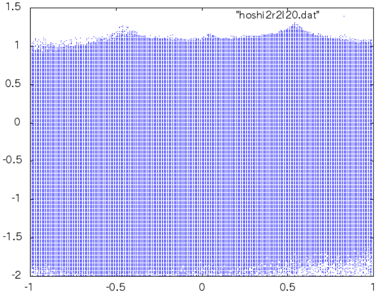
t-imag(minus)

Figure of Jorgensen's inequality,
different up and down
(t-imag(plus), t-imag(minus))

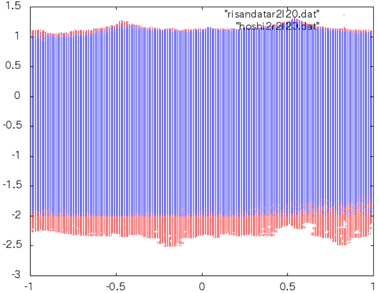
- elliptic ($x=2, y=0.58i+1.69$)



Jorgensen



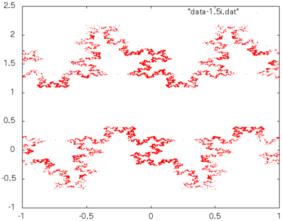
elliptic



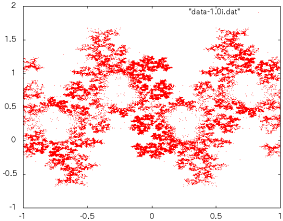
Jorgensen + elliptic

Comparison of
the figure of Jorgensen's inequality
and the figure of elliptic element
: roughly match

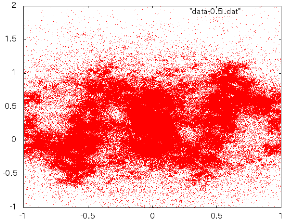
- limit set (x=2,y=0.59i+1.82,z=y+2i)



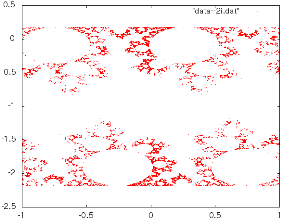
t=1.5i



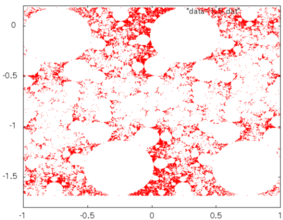
t=1.0i



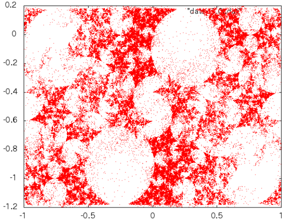
t=0.5i



t=-2.0i



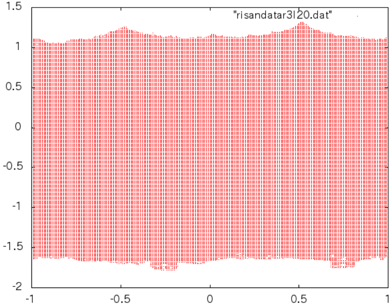
t=-1.5i



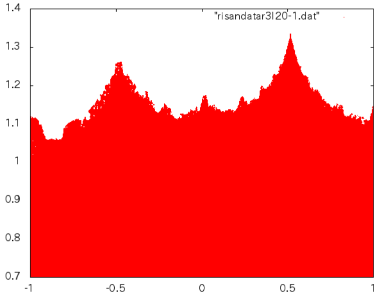
t=-1.0i

The boundary is in the vicinity of +1.0i,-1.5i.

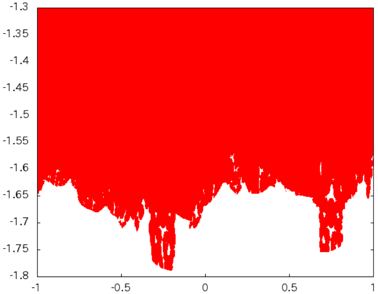
- Jorgensen (x=2,y=0.59i+1.82)



Jorgensen



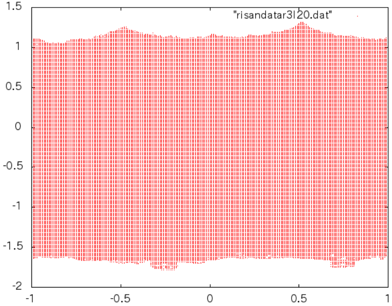
t-imag(plus)



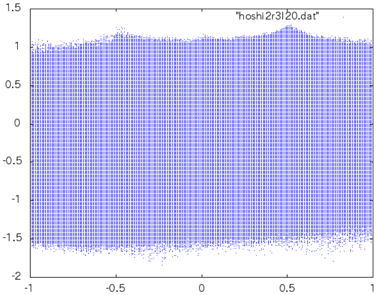
t-imag(minus)

Figure of Jorgensen's inequality,
different up and down
(t-imag(plus), t-imag(minus))

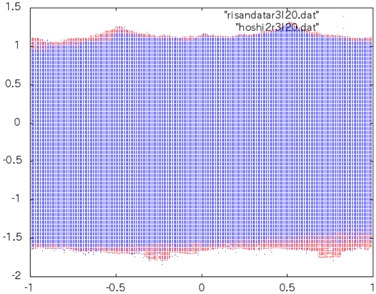
- elliptic (x=2,y=0.59i+1.82)



Jorgensen



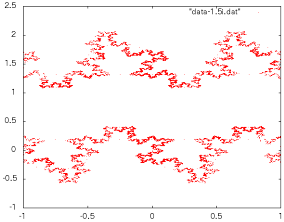
elliptic



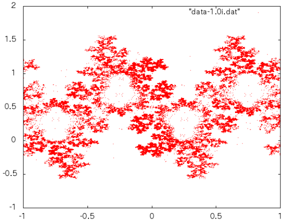
Jorgensen + elliptic

Comparison of
the figure of Jorgensen's inequality
and the figure of elliptic element
: roughly match

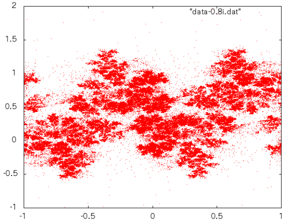
- limit set (x=2,y=0.60i+1.93,z=y+2i)



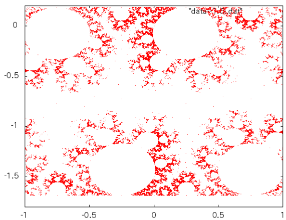
t=1.5i



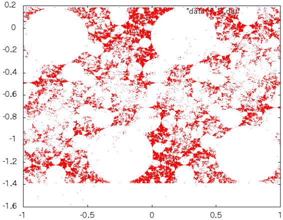
t=1.0i



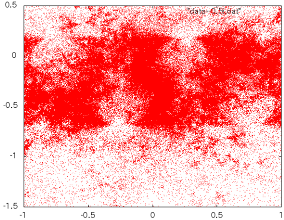
t=0.8i



t=-1.5i



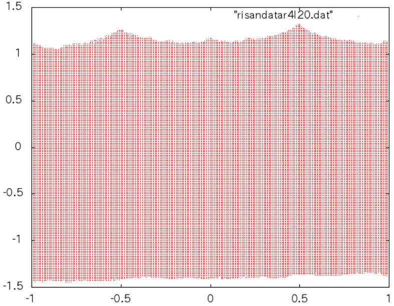
t=-1.2i



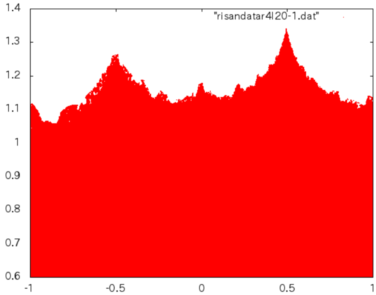
t=-0.5i

The boundary is in the vicinity of +1.0i,-1.2i.

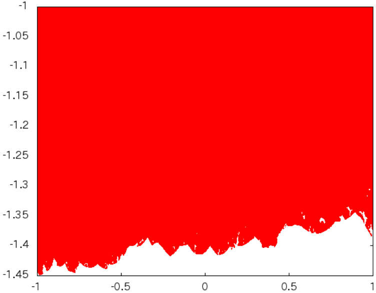
- Jorgensen (x=2,y=0.60i+1.93)



Jorgensen



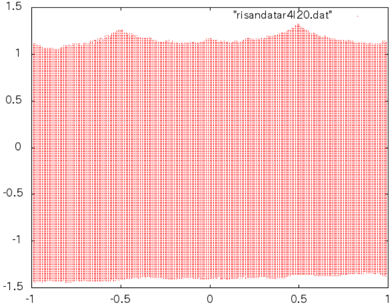
t-imag(plus)



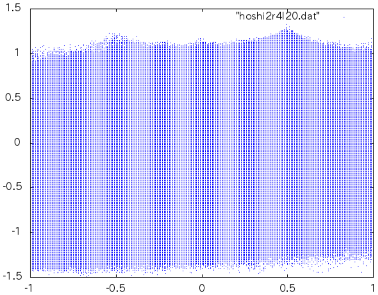
t-imag(minus)

Figure of Jorgensen's inequality,
different up and down
(t-imag(plus), t-imag(minus))

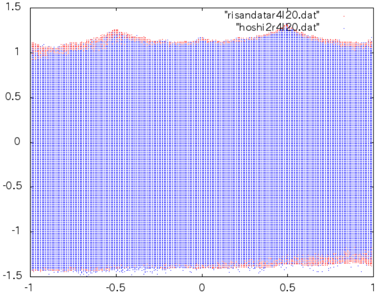
- elliptic (x=2,y=0.60i+1.93)



Jorgensen



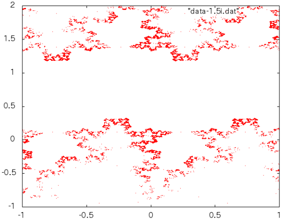
elliptic



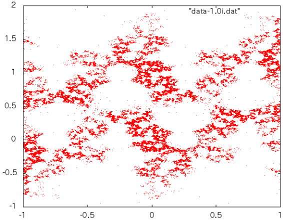
Jorgensen + elliptic

Comparison of
the figure of Jorgensen's inequality
and the figure of elliptic element
: roughly match

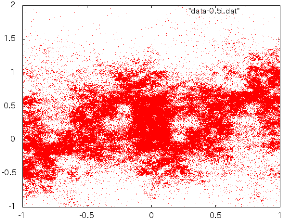
⟨1⟩limit set (x=2,y=1.41i+1.69,z=y+2i)



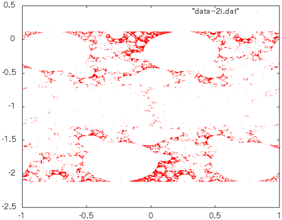
t=1.5i



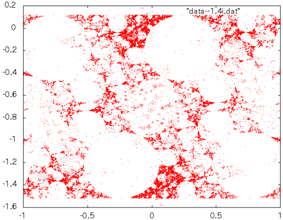
t=1.0i



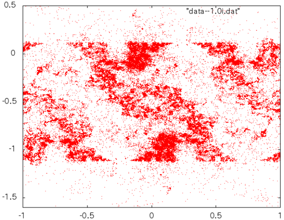
t=0.5i



t=-2.0i



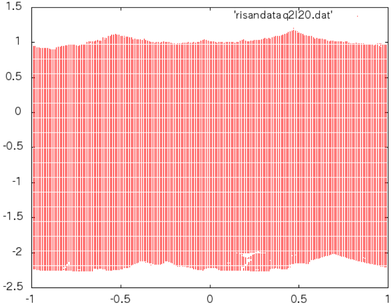
t=-1.4i



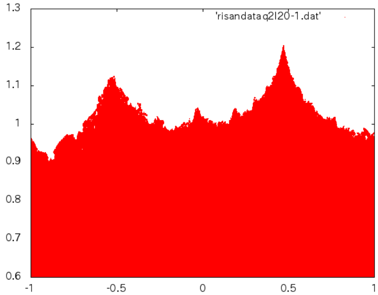
t=-1i

The boundary is in the vicinity of +1.0i,-1.4i.

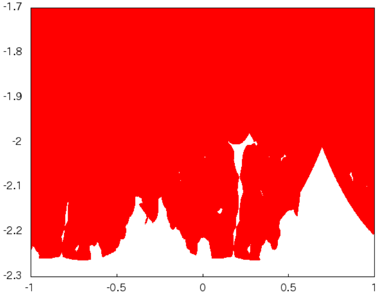
⟨2⟩Jorgensen (x=2,y=1.41i+1.69)



Jorgensen



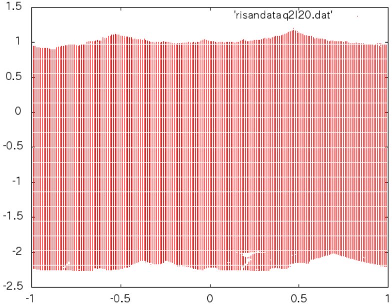
t-imag(plus)



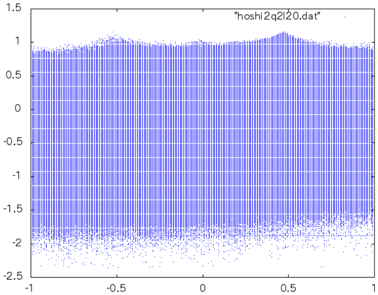
t-imag(minus)

Figure of Jorgensen's inequality,
different up and down
(t-imag(plus), t-imag(minus))

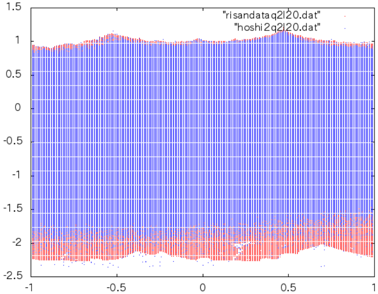
⟨3⟩elliptic (x=2,y=1.41i+1.69)



Jorgensen



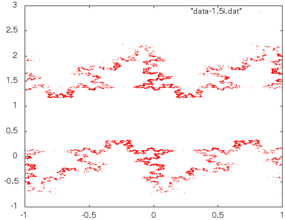
elliptic



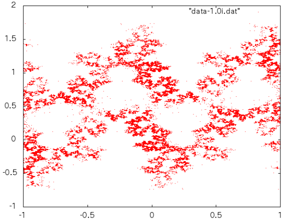
Jorgensen + elliptic

Comparison of
the figure of Jorgensen's inequality
and the figure of elliptic element
: roughly match

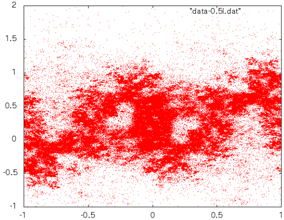
- limit set (x=2,y=1.40i+1.82,z=y+2i)



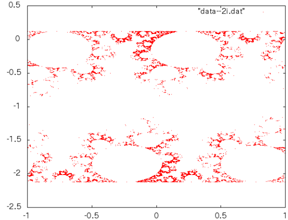
t=1.5i



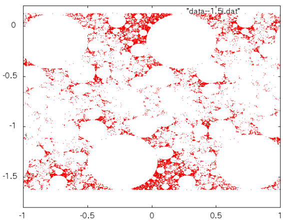
t=1.0i



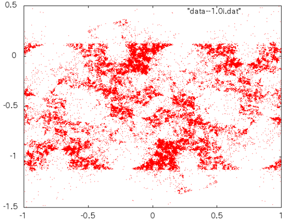
t=0.5i



t=-2.0i



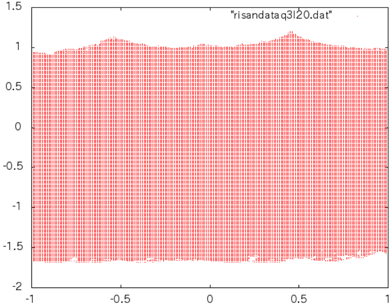
t=-1.5i



t=-1.0i

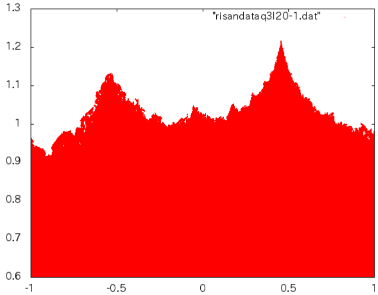
The boundary is in the vicinity of +1.0i,-1.5i.

- Jorgensen $(x=2, y=1.40i+1.82)$

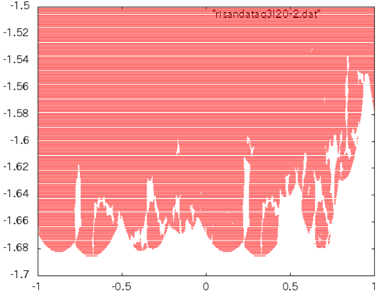


Jorgensen

Figure of Jorgensen's inequality, different up and down (t-imag(plus), t-imag(minus))

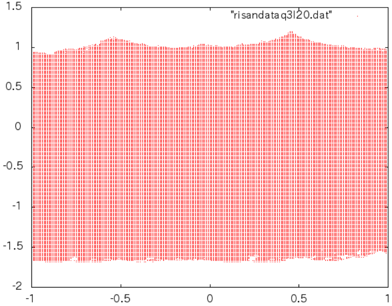


t-imag(plus)

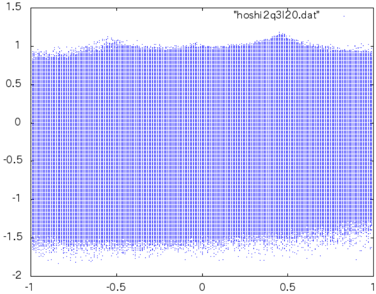


t-imag(minus)

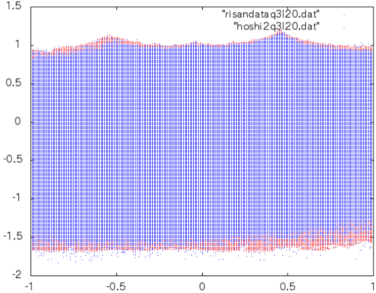
- elliptic (x=2,y=1.40i+1.82)



Jorgensen



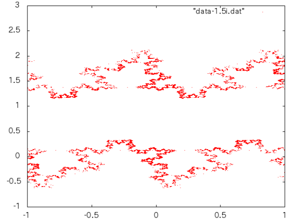
elliptic



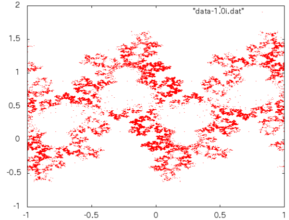
Jorgensen + elliptic

Comparison of
the figure of Jorgensen's inequality
and the figure of elliptic element
: roughly match

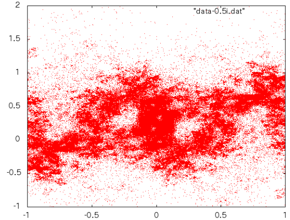
- limit set (x=2,y=1.39i+1.93,z=y+2i)



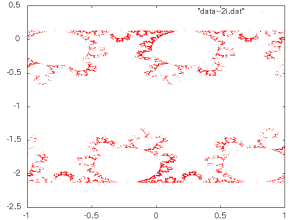
t=1.5i



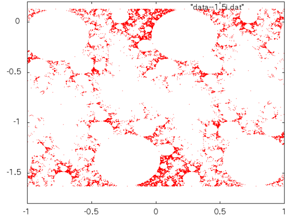
t=1.0i



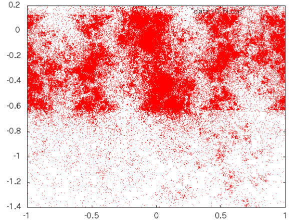
t=0.5i



t=-2.0i



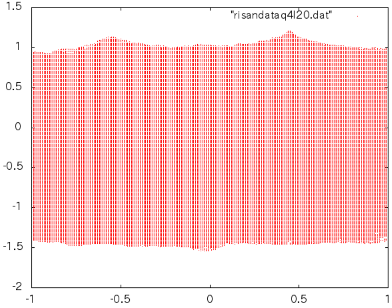
t=-1.5i



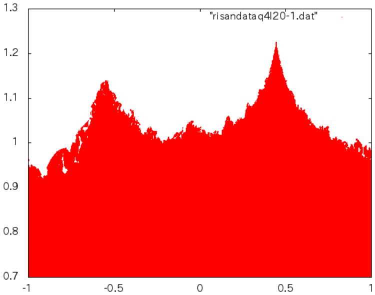
t=-0.5i

The boundary is in the vicinity of +1.0i,-1.5i.

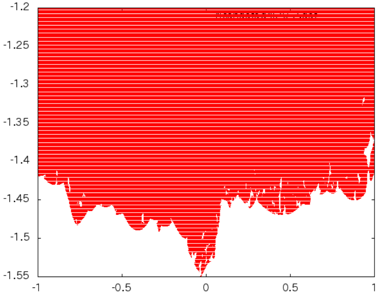
- Jorgensen $(x=2, y=1.39i+1.93)$



Jorgensen



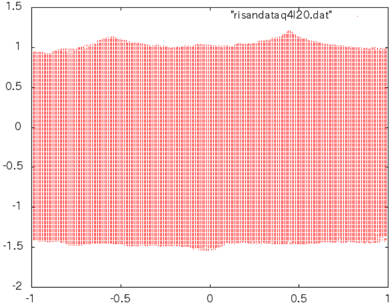
t-imag(plus)



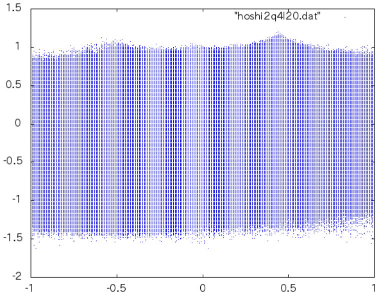
t-imag(minus)

Figure of Jorgensen's inequality,
different up and down
(t-imag(plus), t-imag(minus))

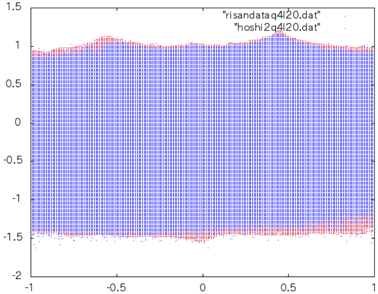
- elliptic (x=2,y=1.39i+1.93)



Jorgensen



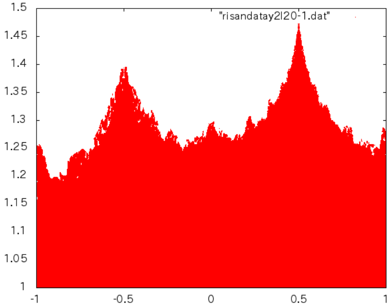
elliptic



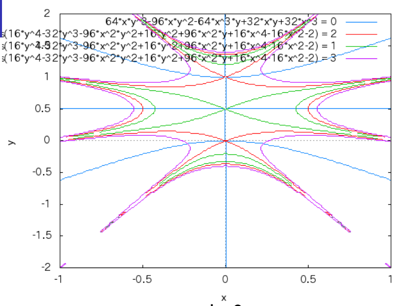
Jorgensen + elliptic

Comparison of
the figure of Jorgensen's inequality
and the figure of elliptic element
: roughly match

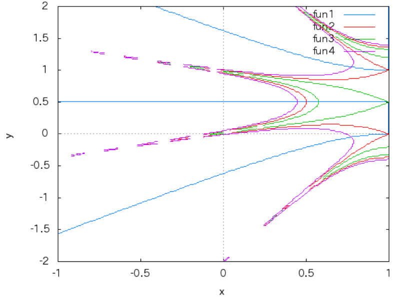
⟨4⟩real loci (x=2,y=2,z=2+2i)



Jorgensen



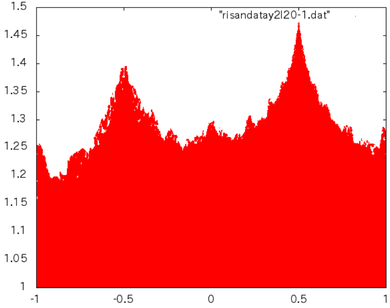
treal=0



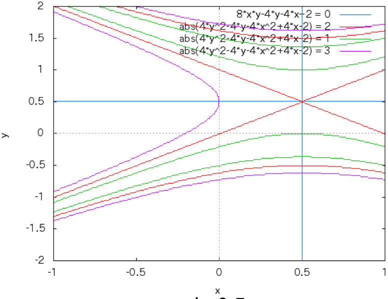
treal=1

Figure of real loci
in case the real part of t is 0 and 1
: almost coincides with Jorgensen's
inequality

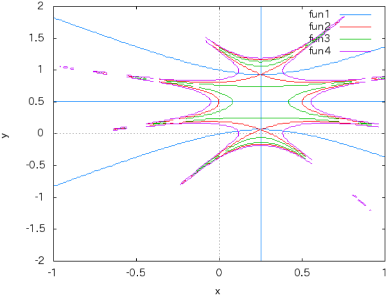
⟨4⟩real loci (x=2,y=2,z=2+2i)



Jorgensen



treal=0.5



treal=0.25

Figure of real loci in case
the real part of t is 0.5 and 0.25
: almost coincides with Jorgensen's
inequality

- Patterns of "highest words"

In many cases, there are combinations of commutators and conjugates.

• t=0	• t=1	• t=0.5
acBAbC	CABabacA	adcBAbaD
aDcdAC	CABAbaca	aDcdABab
aDCdAc	CACDcdca	ABAbaDCd
AcaDCd	CDcdcaCA	AdABabCD
BabCac	acACDCdc	baDCdcBc
⋮	⋮	⋮
⋮	⋮	⋮

$A = g_{y,t}(X),\ B = g_{y,t}(Y),\ C = g_{y,t}(Z),\ D = g_{y,t}(W)\ \ a = A^{-1},\ b = B^{-1},\ c = C^{-1}, d = D^{-1}$

$\langle example \rangle$ commutator : ABab,CDcd$\cdots$ / conjugate : Aba,Dad$\cdots$

Consideration

D_J: the discrete subset given by Jorgensen's inequality
D_E: the discrete subset given by elliptic elements

- For various parameters,we draw limit sets.
- D_J has a rotational symmetry of order 2.(y=2)
- D_J and D_E roughly coincide.

Future tasks

・It is necessary to increase the length of the matrix product of Möbius groups.

・Examine the real loci according to parameters other than y=2.

・From a character string to be real loci,examine whether there is a rule.

References

[1] David Mumford,Caroline Series,David Wright,"INDRA'S PEARLS-The vision of feilx klein",Cambridge University Press 2002.

[2] Linda Keen,Nikola Lakic,"Hyperbolic Geometry from a Local Viewpoint",London Mathematical Society Student Texts 68 2007.

[3] Sara Maloni,Caroline Series,"Top terms of polynomial traces in Kra's plumbing construction.