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Topology and Computer 2015@Nihon Univ.

Rigorous numerics for fast-slow systems

Topological shadowing approach

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Dynamical Systems

Time evolution of solution orbits for

$$\dot{x} = f(x) \quad f : \mathbb{R}^n \to \mathbb{R}^n, \text{smooth}$$ or

$$x_{n+1} = f(x_n) \quad f : \mathbb{R}^n \to \mathbb{R}^n, \text{diffeo.}$$

General theory from differential equations …

- Existence
- Uniqueness
- Continuous dependence on initial points

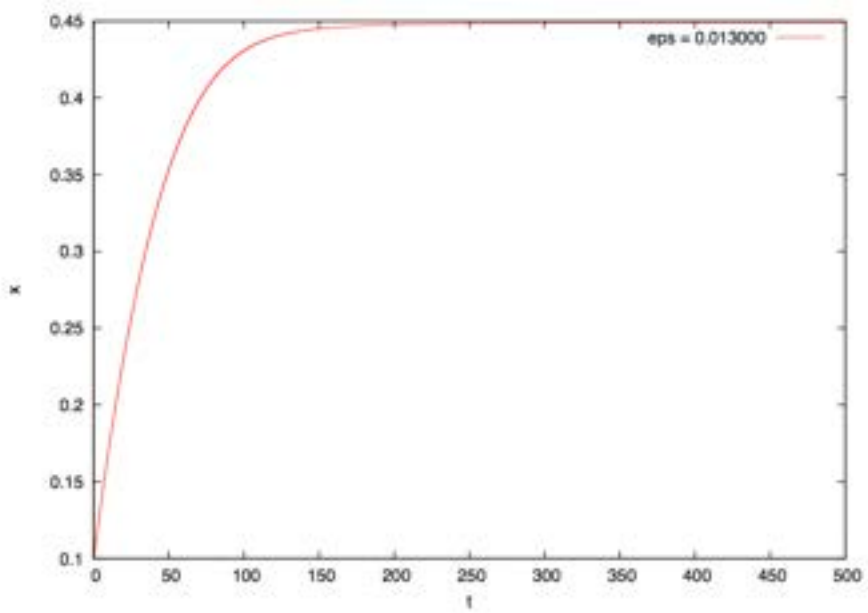
→ **flow**

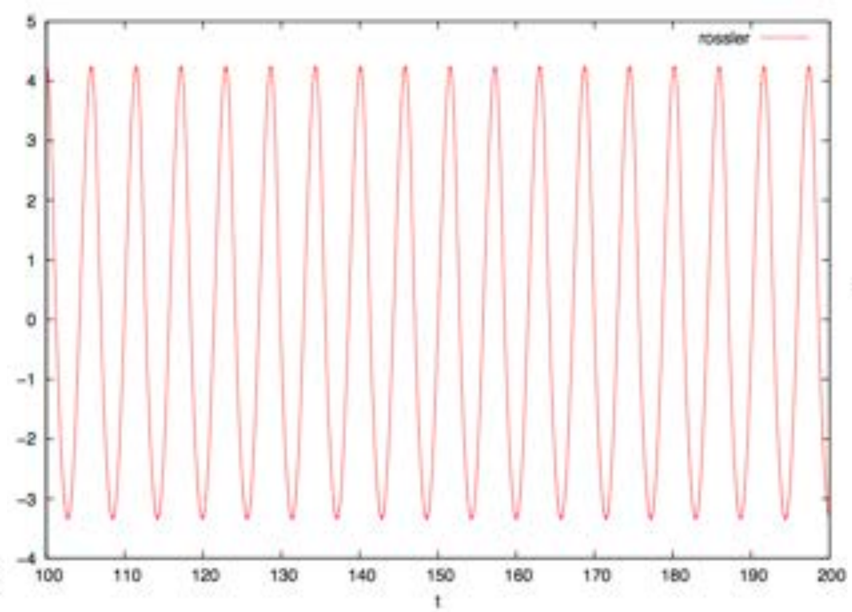
$$\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \text{smooth s.t.}$$

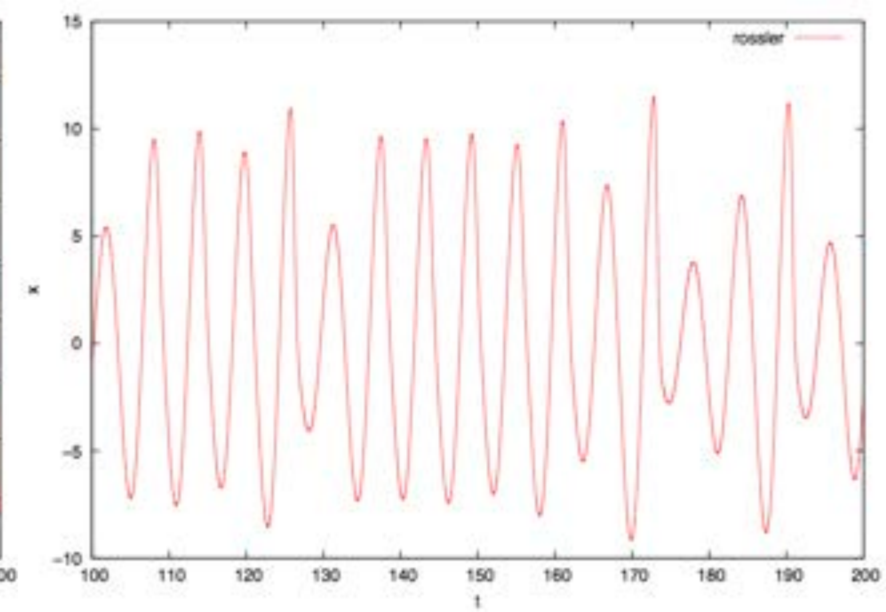
$$\varphi(0, x) = x,\ \ \varphi(t+s, x) = \varphi(t, \varphi(s, x))$$

Key Task 1 : Asymptotic Behavior

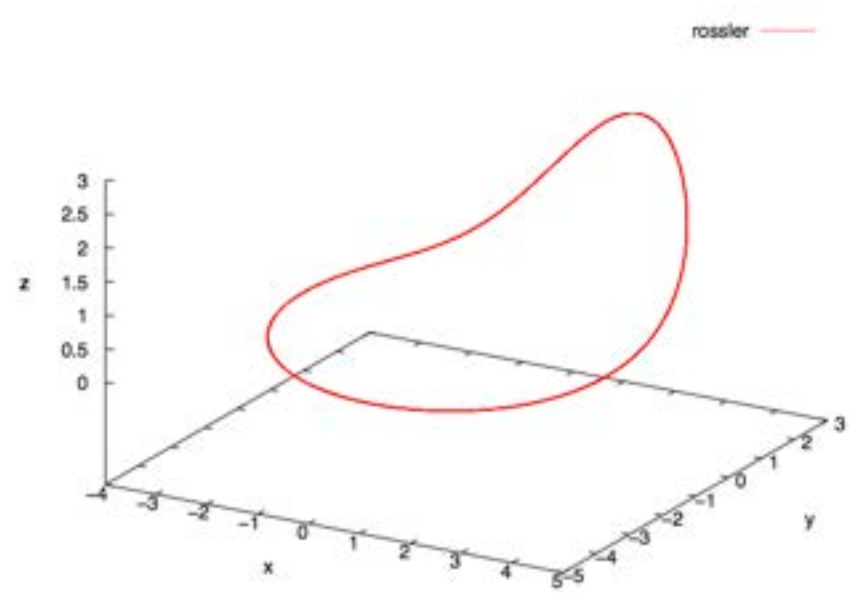
Typical examples. top : (t,x)-plot, bottom : (x,y,z)-plot

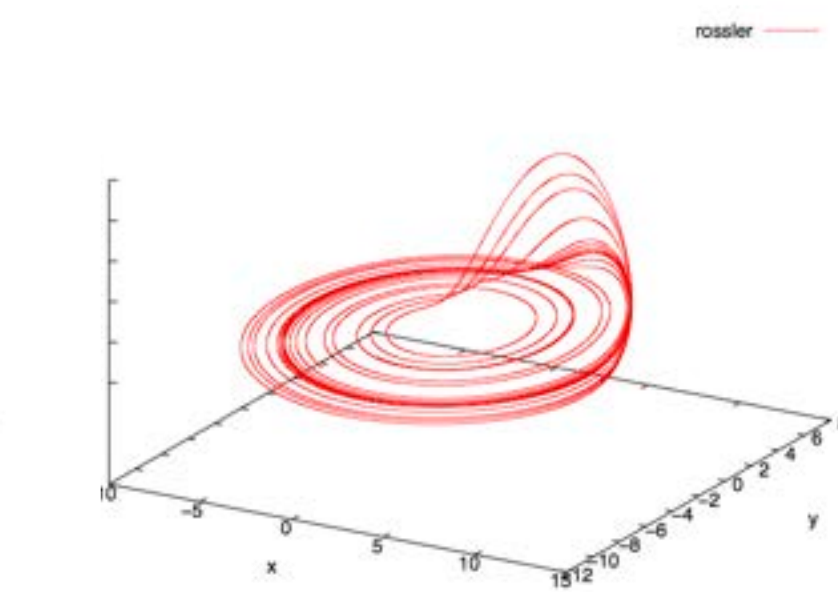






$$f(x) = 0$$



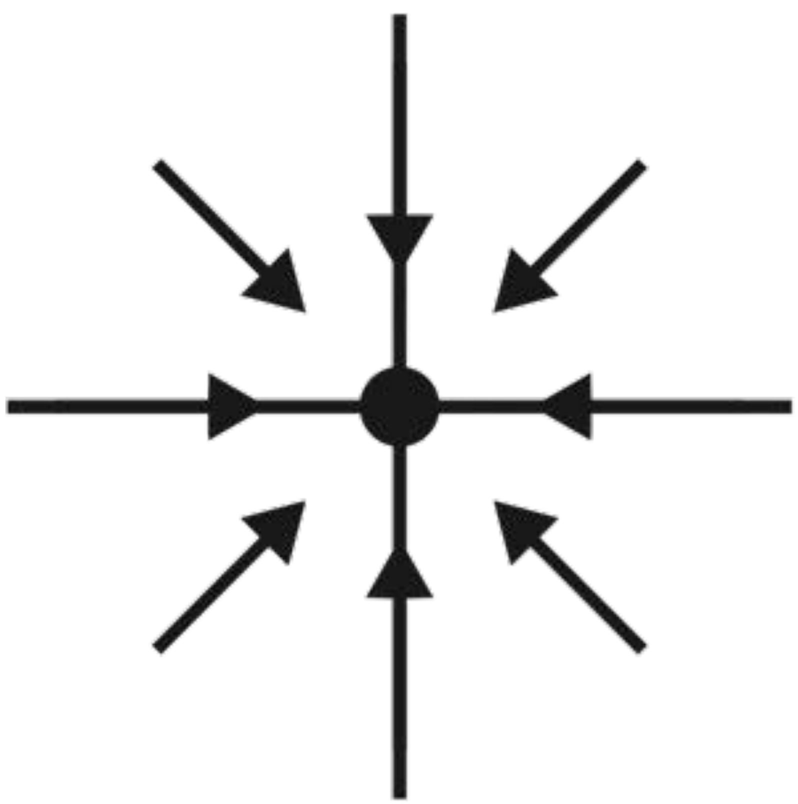


Equilibrium   Periodic orbit   Chaos

Key Task 2 : Stability

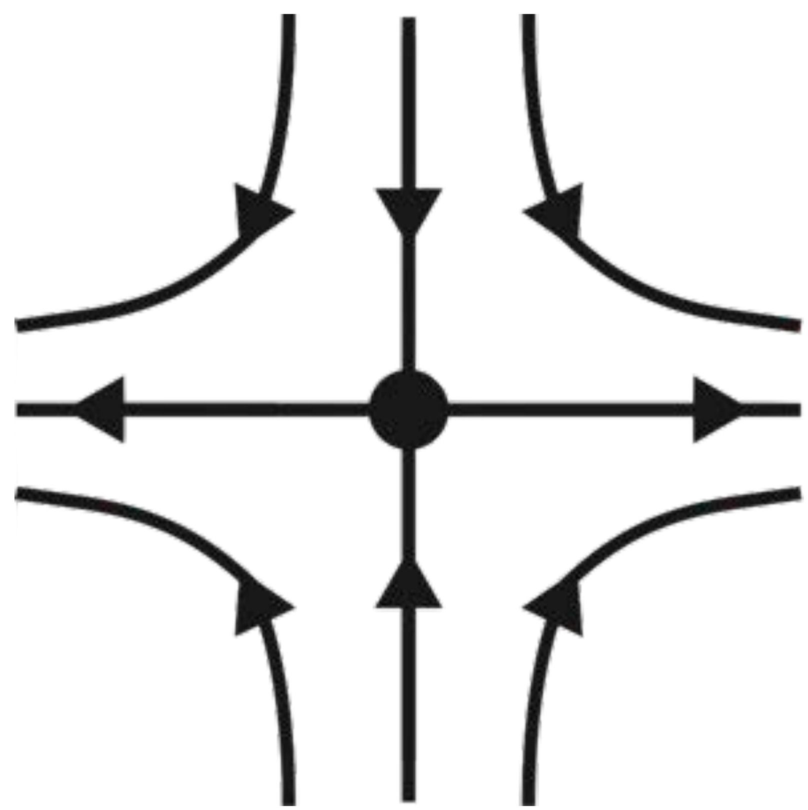
Typical examples. stability of equilibria

$$\dot{x} = ax \ (a < 0)$$
$$\dot{y} = by \ (b < 0)$$



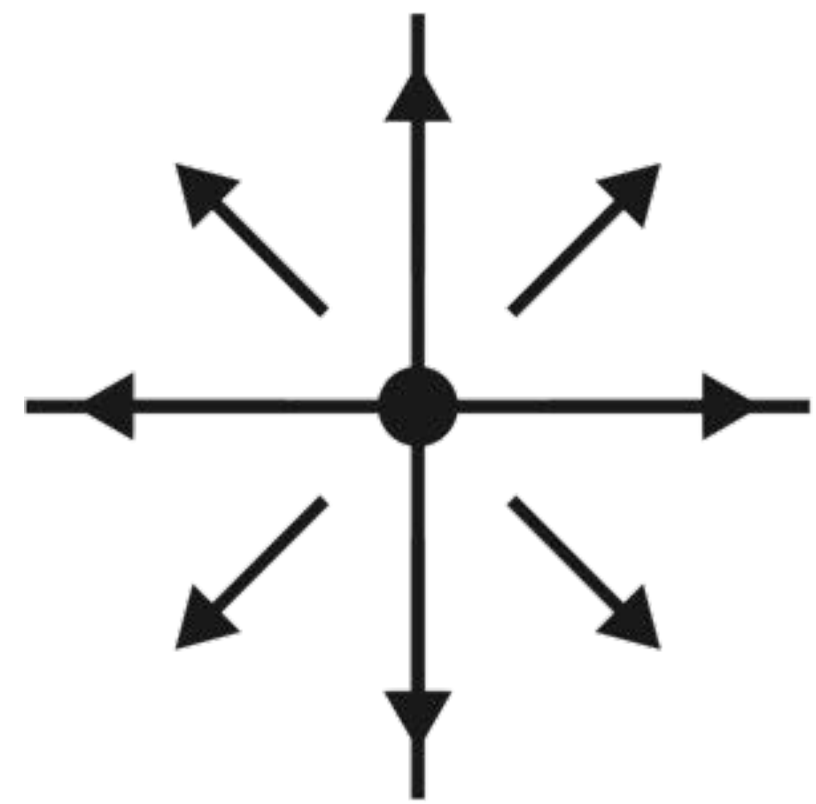
Sink
(stable)

$$\dot{x} = ax \ (a > 0)$$
$$\dot{y} = by \ (b < 0)$$



Saddle
(unstable)

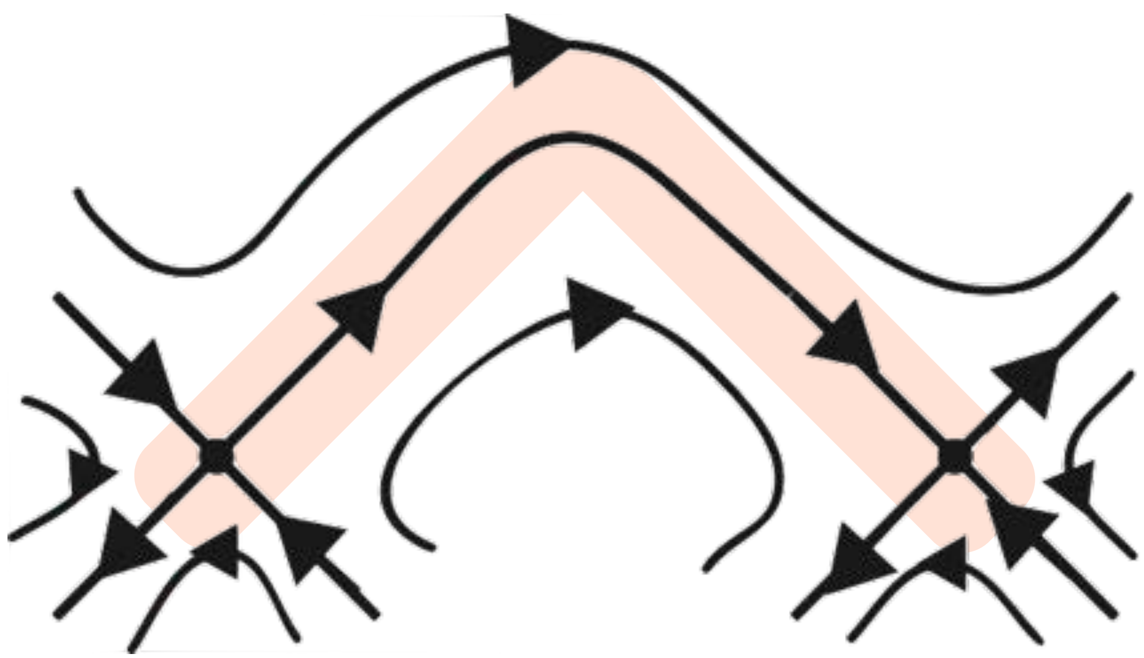
$$\dot{x} = ax \ (a > 0)$$
$$\dot{y} = by \ (b > 0)$$



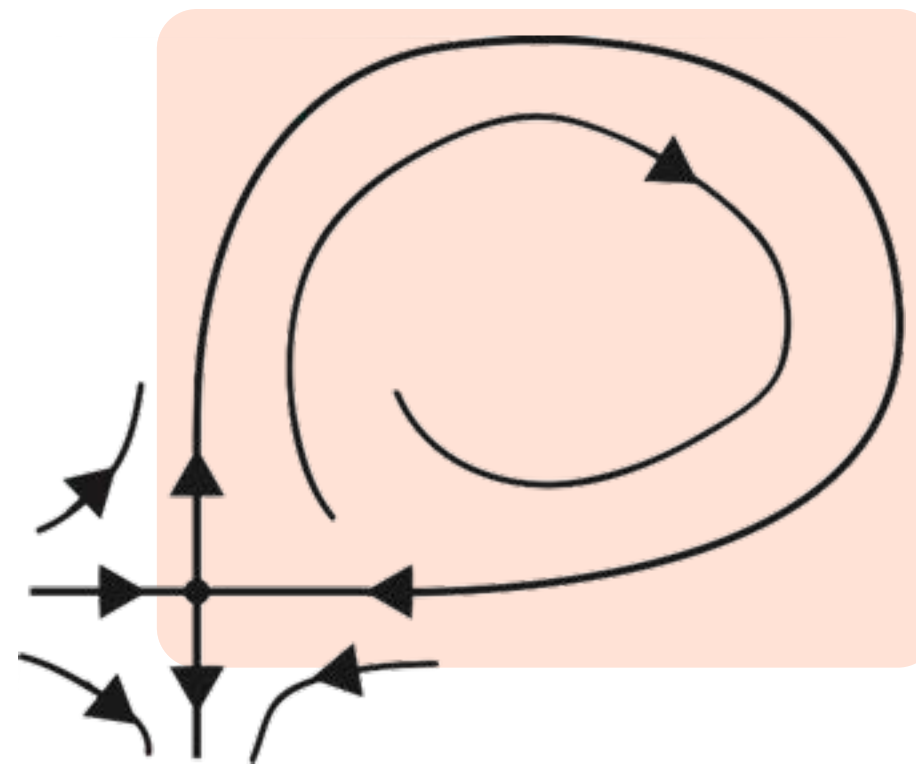
Source
(unstable)

Key Task 3 : Global Dynamics

Typical examples.



Heteroclinic orbit



Homoclinic orbit

Global orbits containing **unstable** equilibria → Global bifurcation, Chaos and more complex phenomena

Fast-slow system

$$(*)_\epsilon \qquad \begin{aligned} \dot{x} &= f(x, y, \epsilon) \\ \dot{y} &= \epsilon g(x, y, \epsilon), \quad 0 \leq \epsilon \ll 1 \end{aligned}$$

$$x \in \mathbb{R}^n \, : \, \text{fast}, \; y \in \mathbb{R}^l \, : \, \text{slow}, \; t \in \mathbb{R} \, : \, \text{time}$$

ε ：multiple time-scale parameter

Ex. Traveling wave solutions of reaction-diffusion systems

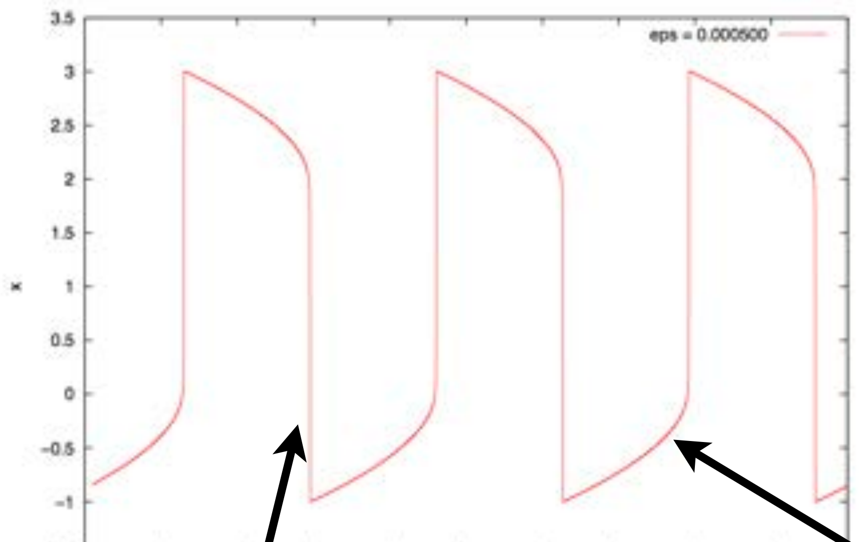
ex. FitzHugh-Nagumo

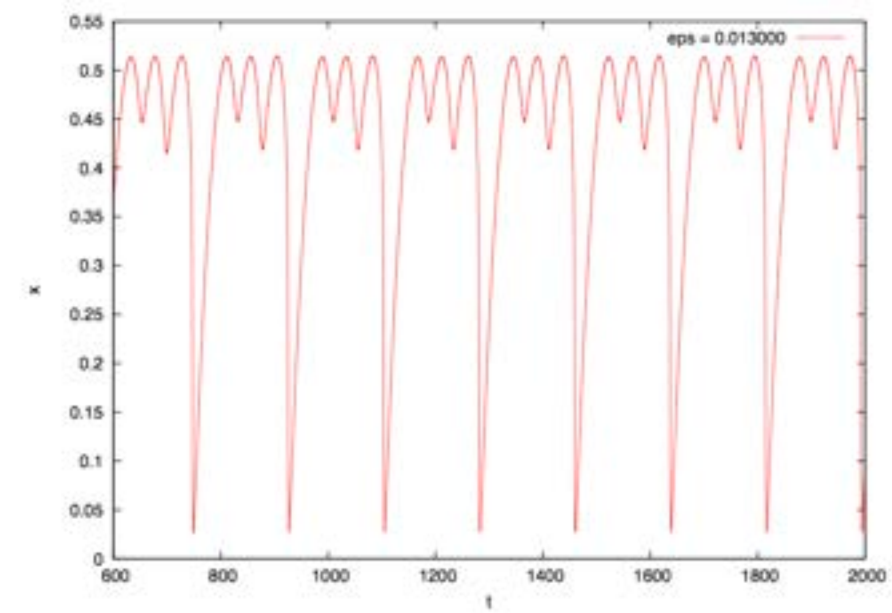
$$\begin{aligned} u_t &= \delta u_{xx} + f(u) - \lambda \\ \lambda_t &= \epsilon(u - \gamma\lambda) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \dot{u} &= v \\ \dot{v} &= \delta^{-1}(\theta v - f(u) + \lambda) \\ \dot{\lambda} &= \epsilon\theta^{-1}(u - \gamma\lambda) \end{aligned}$$

$$u(x, t) \mapsto u(x - \theta t)$$

Fast-slow system

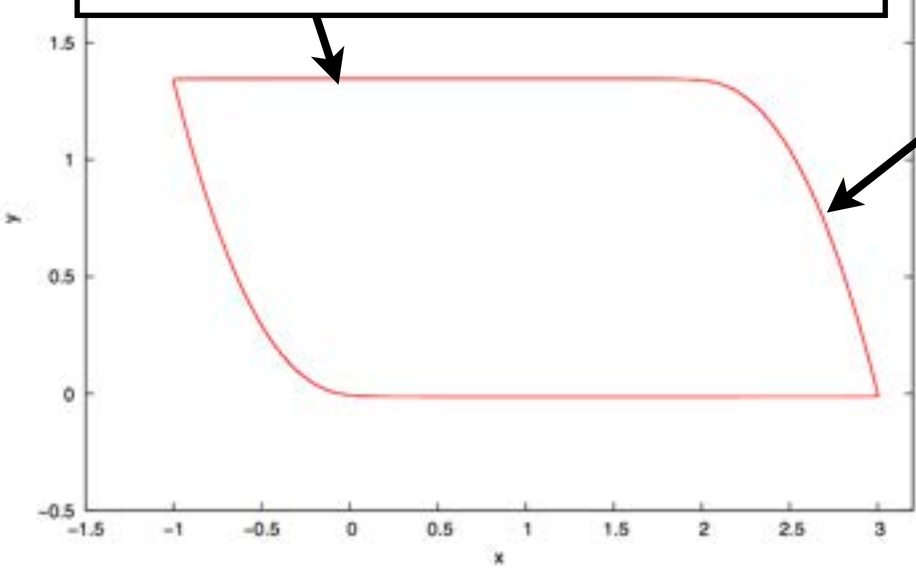
Typical examples. top : (t,x)-plot, bottom : (x,y)-plot

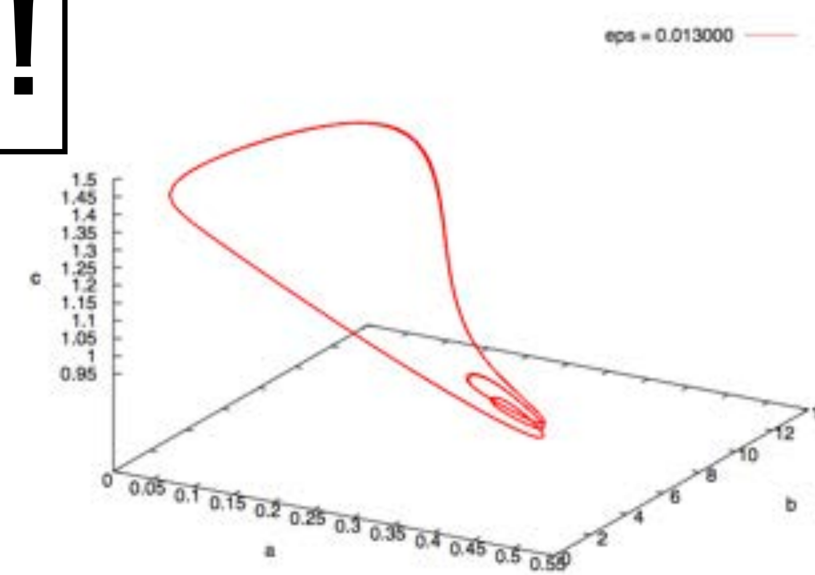




Very fast !

Very slow !

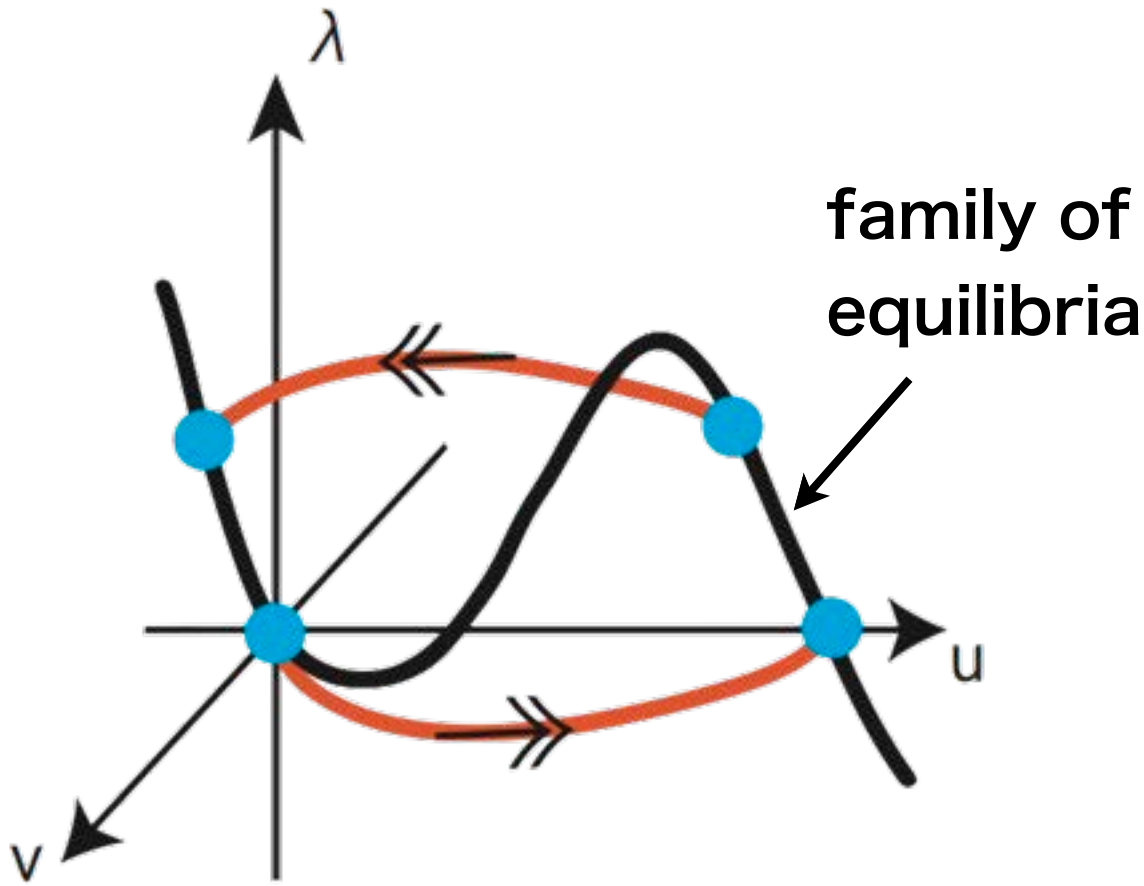


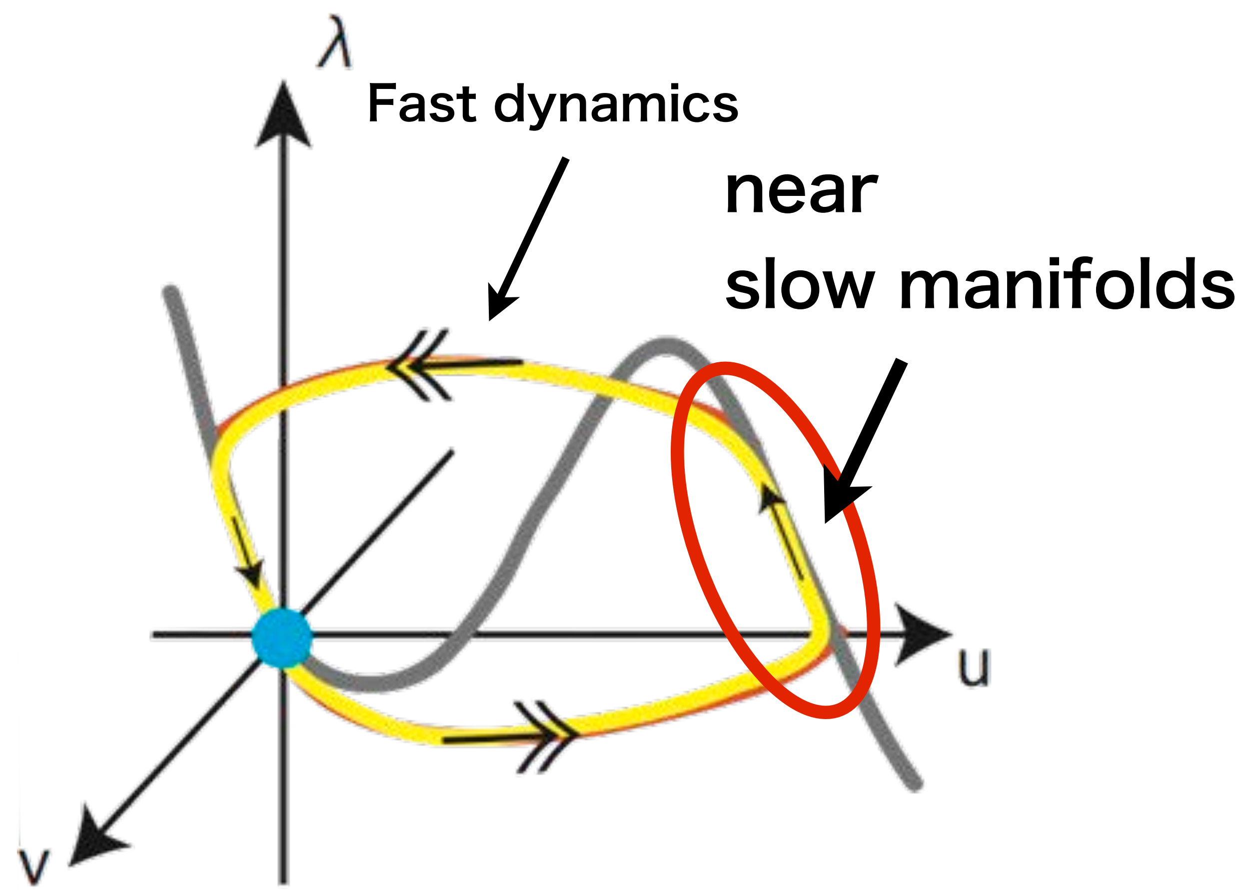


Relaxation Oscillation

Mixed Mode Oscillation

Fast-slow system



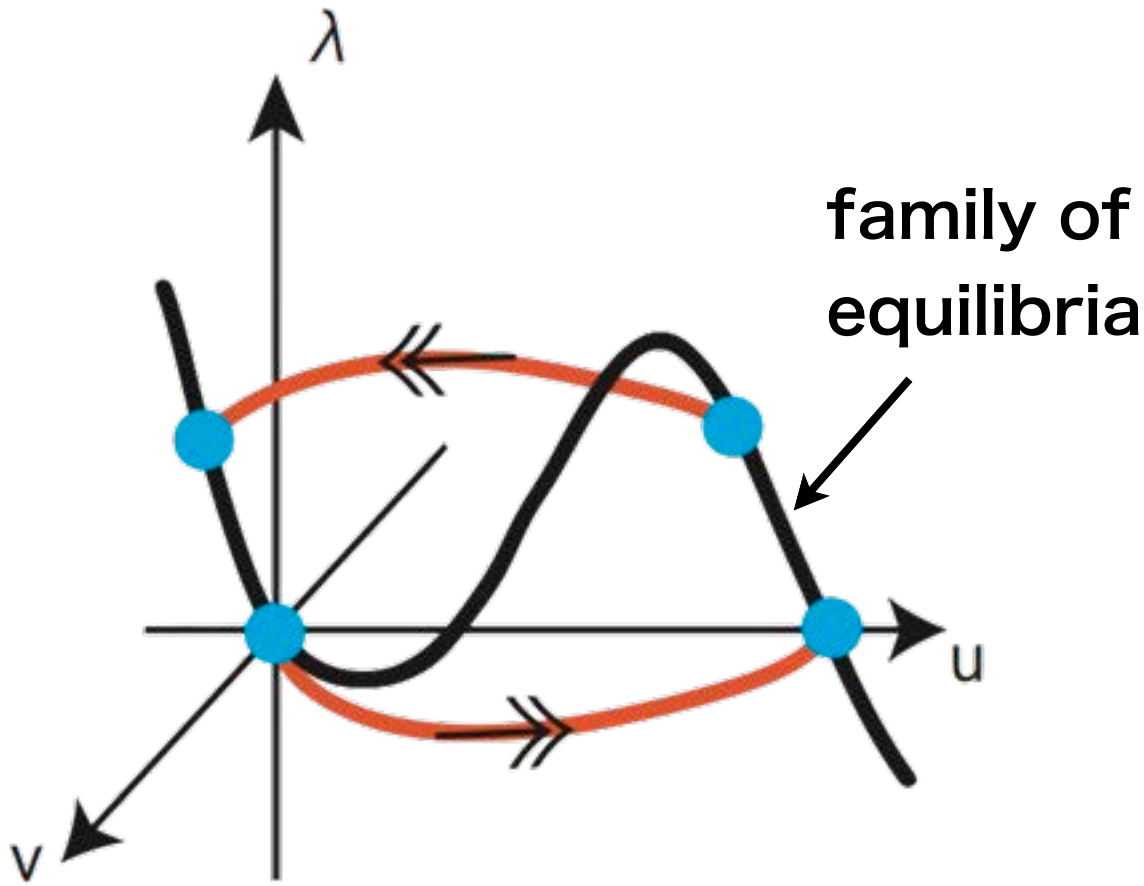


$\varepsilon = 0$ $\varepsilon > 0$: Sufficiently Small

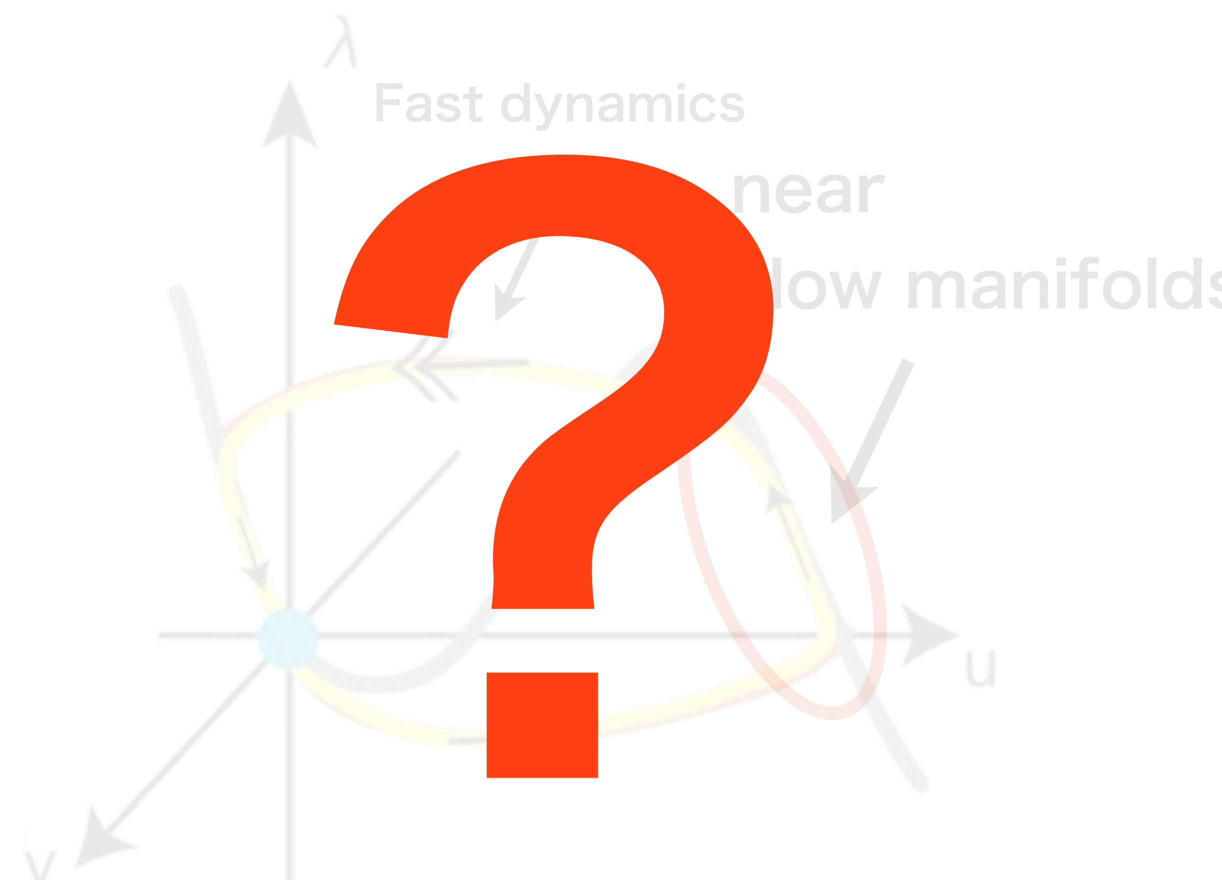
Singular Perturbation Method

- Analytic
- Geometric

Fast-slow system



$\varepsilon = 0$



$\varepsilon > 0$: Given

near “slow manifolds” …

- Speed of trajectories is **very slow !**
- **Need control of instability.**

How do we calculate solutions of such systems with mathematical rigor ?

Analytic Approach (typical)

"Reduce problems for solutions to fixed point problems."

Numerical Simulations

"Solve initial value problems or boundary value problems directly."

“See not objects themselves,
but their neighborhoods.”

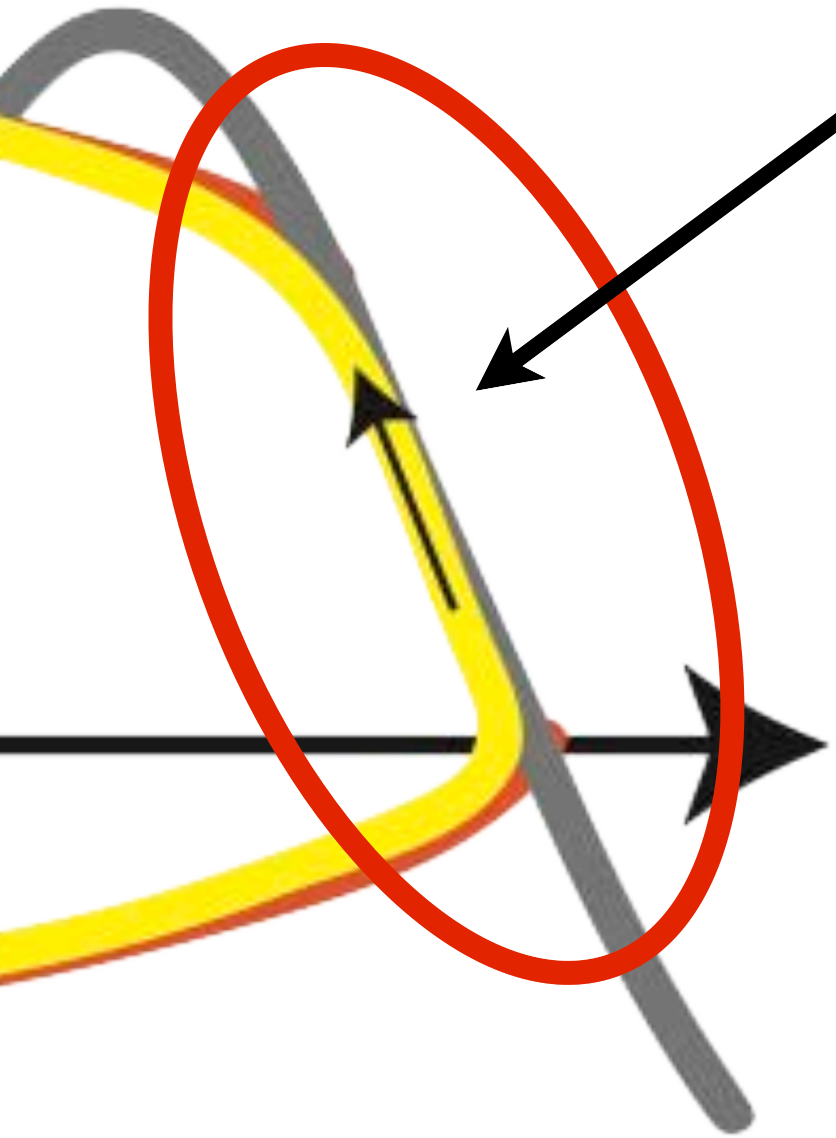
Topological Approach

+ Rigorous Numerics

1. Slow Manifolds

Our desire :

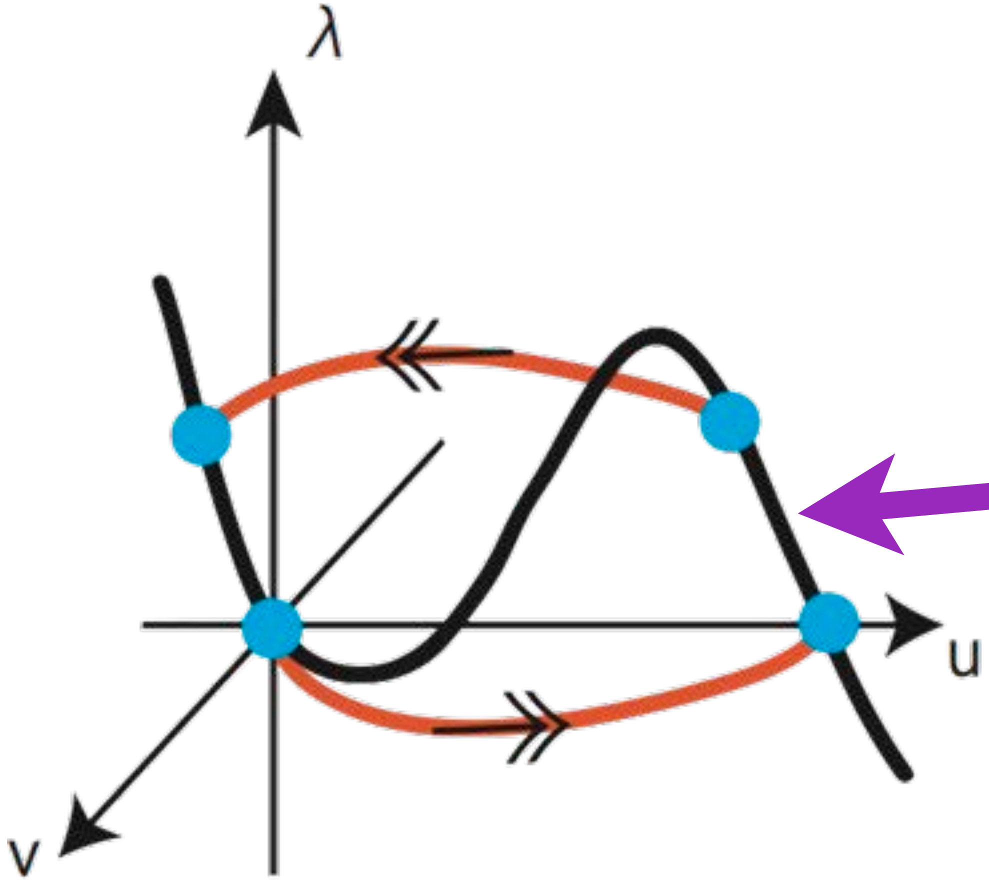
trajectories near “invariant manifolds”



What is this ?

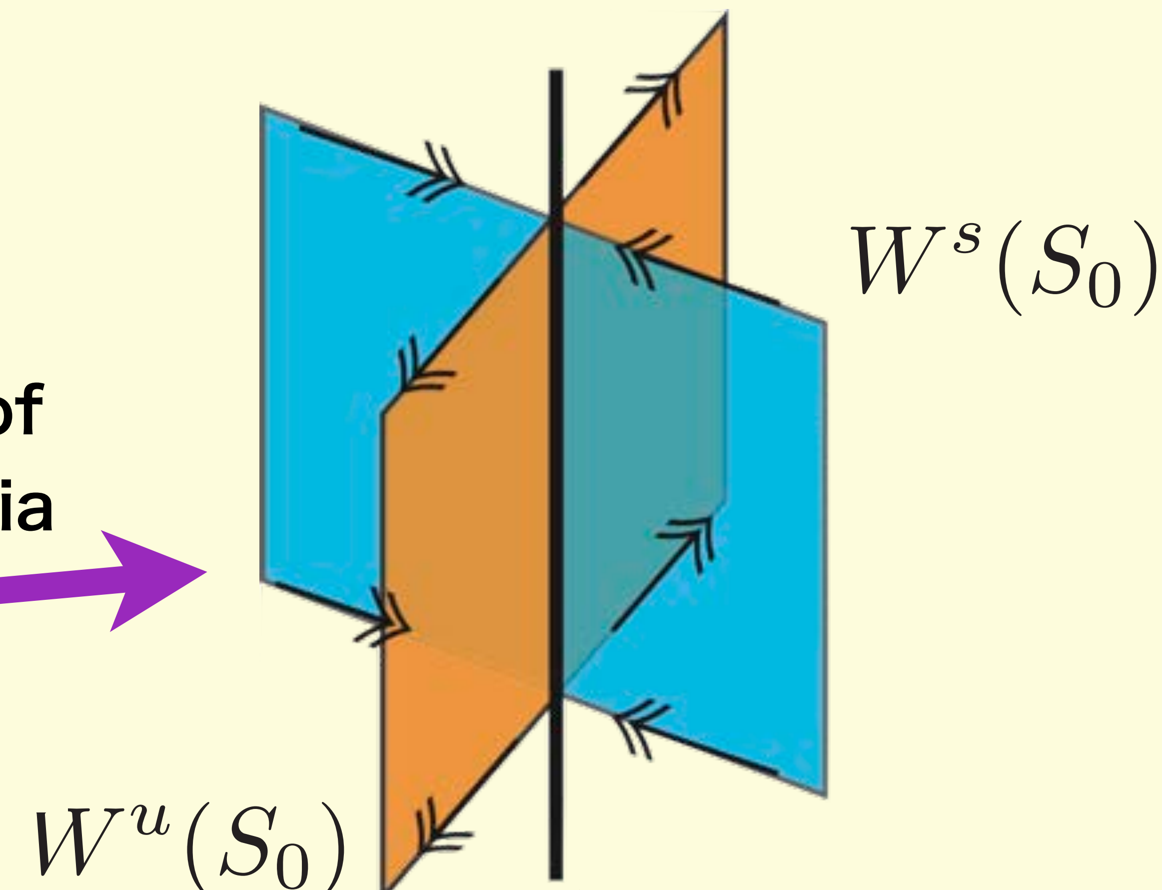
Slow manifold

$$\dot{x} = f(x, y, 0)$$
$$\dot{y} = 0$$



$\varepsilon = 0$

family of equilibria



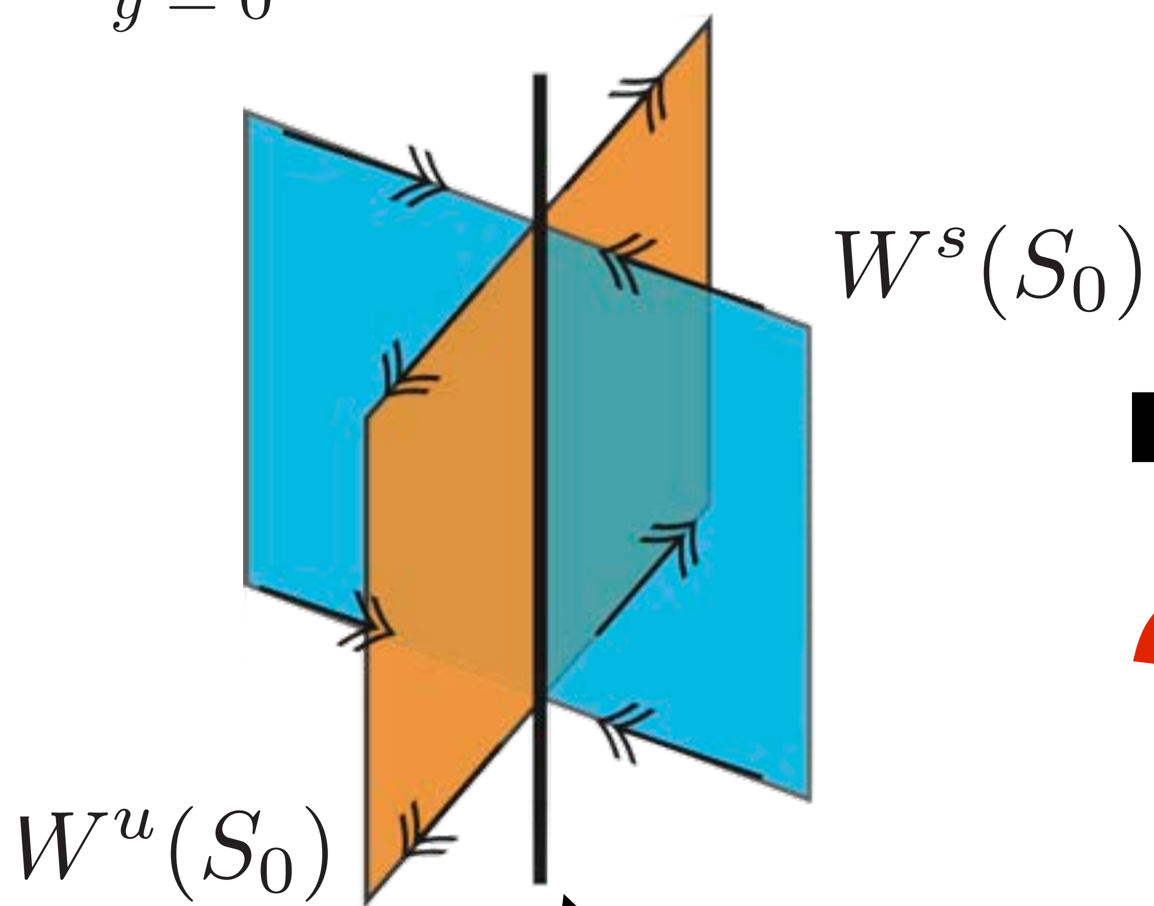
$$S_0 \subset \{f(x, y, 0) = 0\}$$

(invariant)

Slow manifold

$\varepsilon = 0$

$$\dot{x} = f(x, y, 0)$$
$$\dot{y} = 0$$



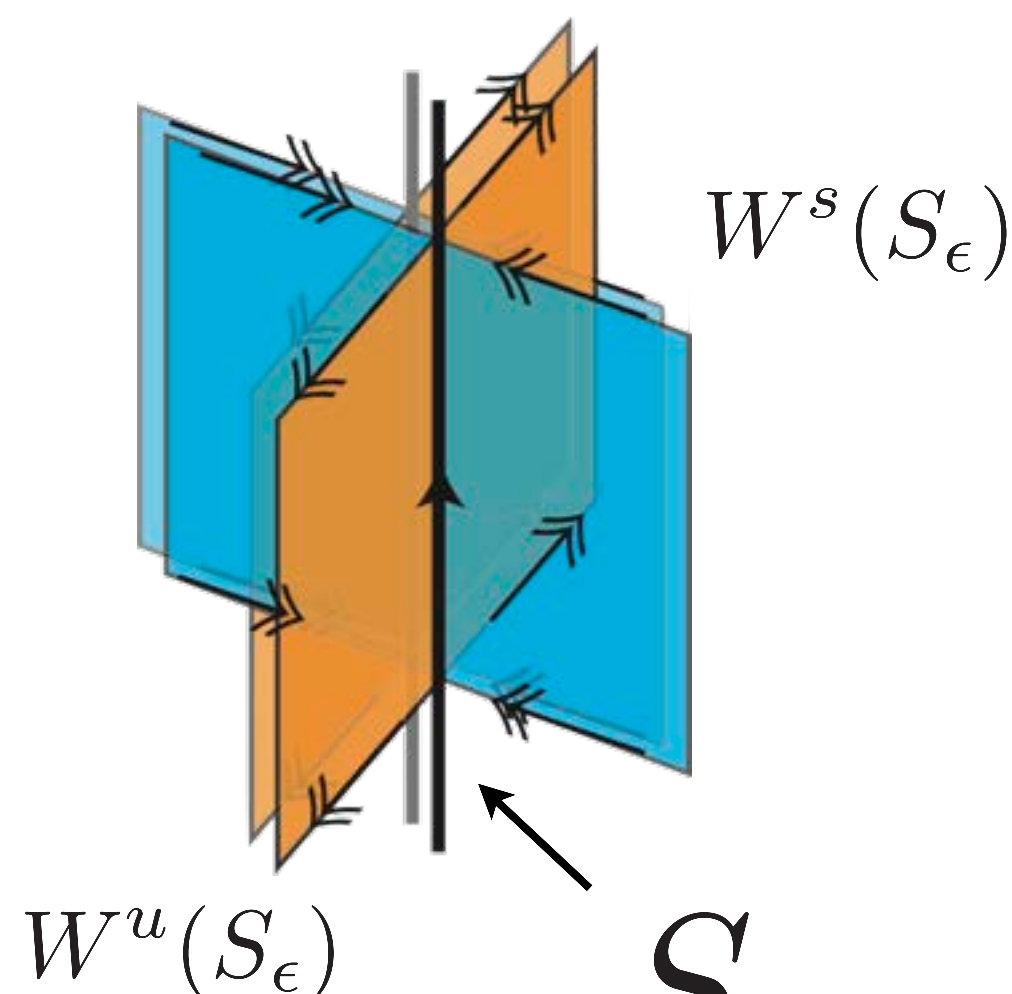
$$S_0 \subset \{f(x, y, 0) = 0\}$$

(invariant)

?

$\epsilon \in (0, \epsilon_0]$

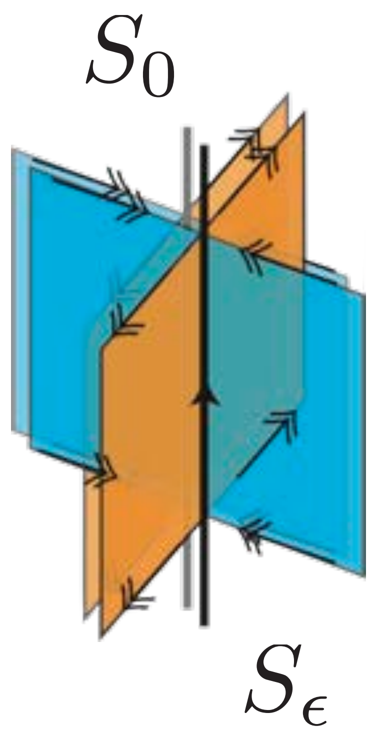
$$\dot{x} = f(x, y, \epsilon)$$
$$\dot{y} = \epsilon g(x, y, \epsilon)$$



S_ϵ

(locally invariant)

Slow manifold



Geometric Singular Perturbation Theory

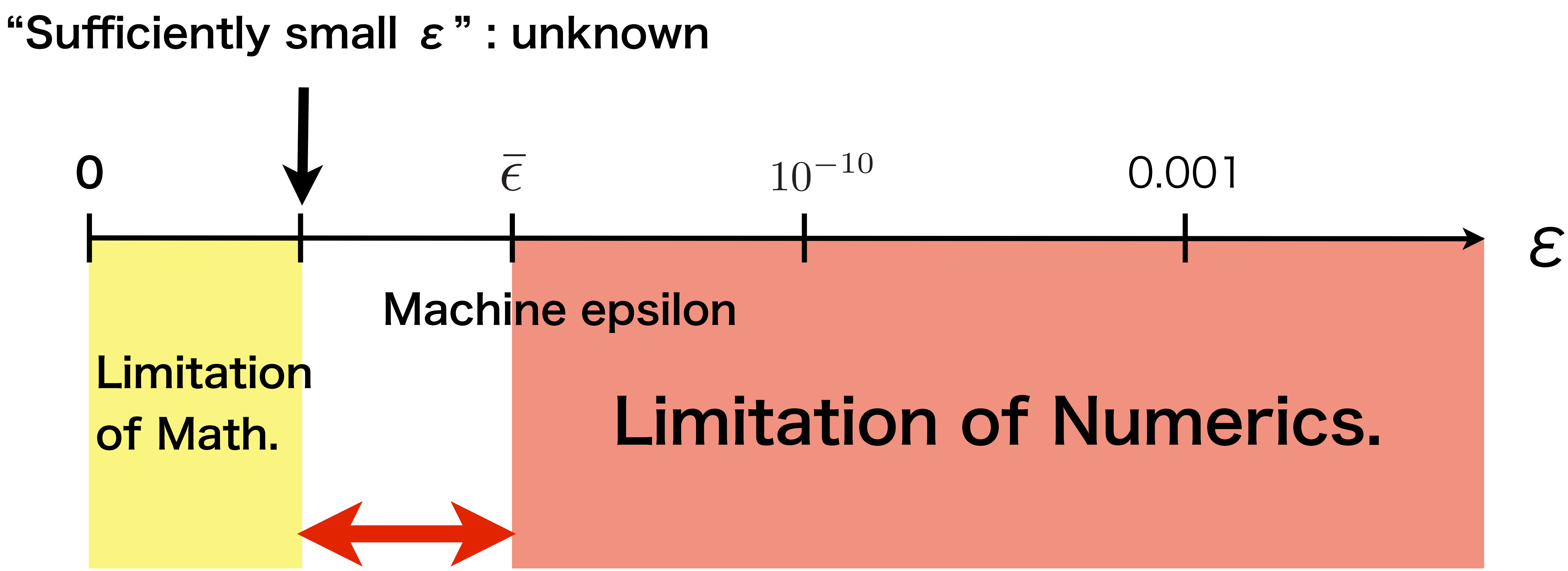
[Fenichel (1979), cf. Jones (1995)]

Persistence of (locally) invariant manifolds for all **sufficiently small ε** under **normal hyperbolicity**

Q. How large is this ε ?

Q. Can we compute solutions for extremely small ε ?

Gap between mathematics and numerics



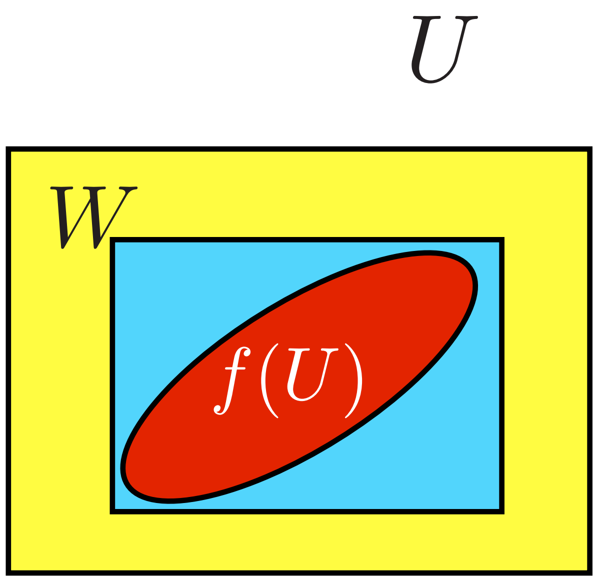
Unavoidable Gap !!

Rigorous Numerics

$\sqrt{2} = 1.41421356$: Not rigorous → cause small errors

$\sqrt{2} \in [1.41421356, 1.41421357]$: enclose errors → Rigorous !

"Interval Arithmetics"



$$X * Y = \{x * y \mid x \in X, y \in Y\} \quad (* = +, -, \times, /)$$

$$f(U) := \{f(x) \mid x \in U\} \subset W$$

Ex.

$U \subset \mathbb{R}^n$: compact, convex s.t. $\Rightarrow \exists x^* \in U \text{ s.t. } f(x^*) = x^*.$

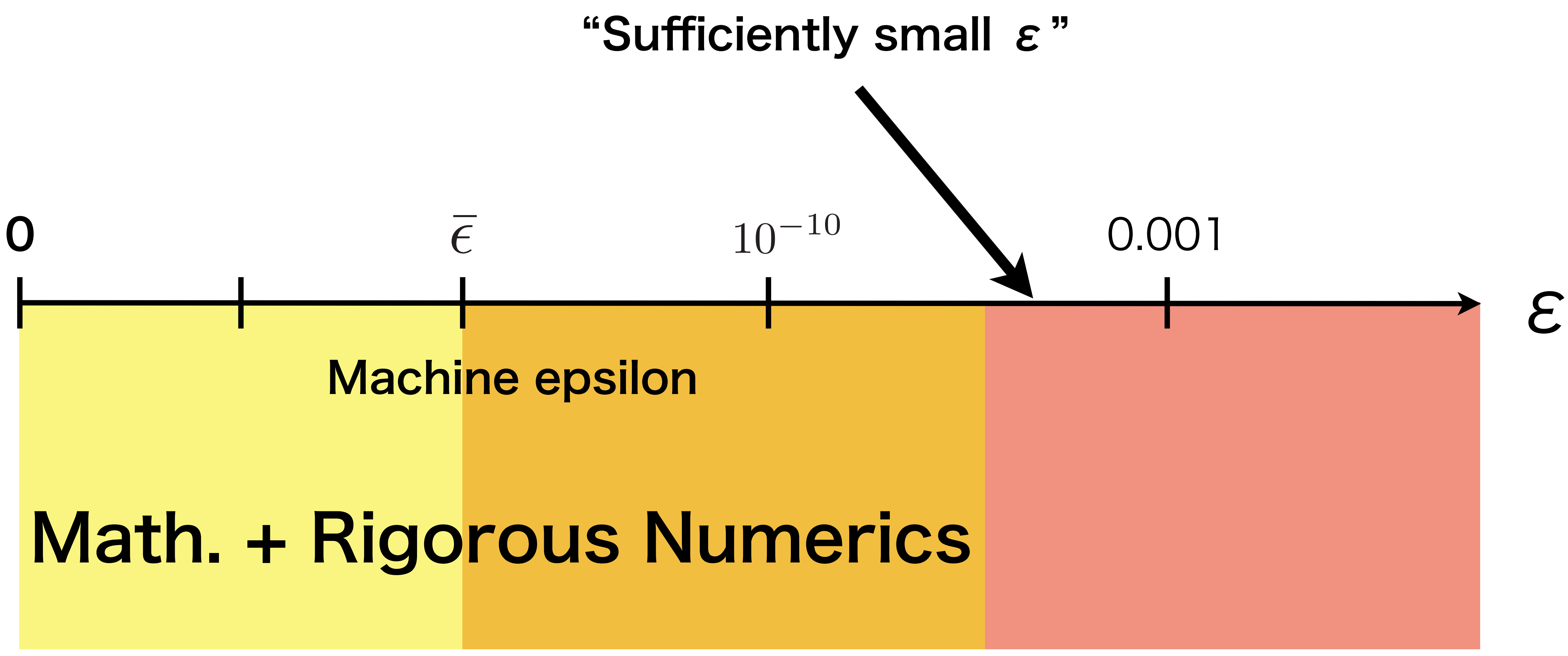
$f(U) \subset U$

U represents x^* !

Enclose all numerical errors

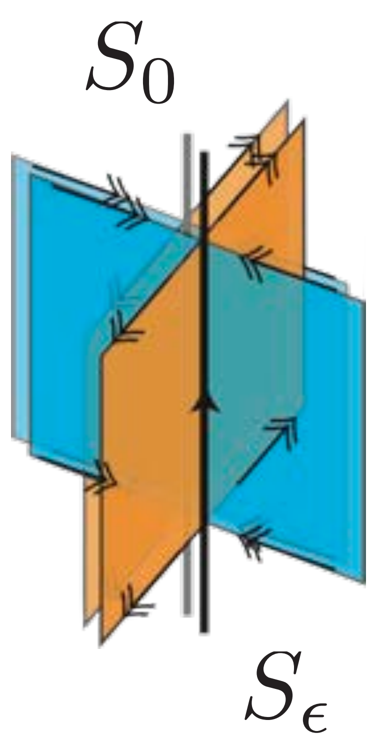
→ Mathematically rigorous numerical results !

Gap between mathematics and numerics



Explicit and Rigorous Coverage !

Slow manifold : topological approach



Geometric Singular Perturbation Theory

[Fenichel (1979), cf. Jones (1995)]

Persistence of (locally) invariant manifolds for all **sufficiently small ε** under **normal hyperbolicity**

diagonalize

$$\dot{a} = Aa + F_1(a,b,y,\epsilon)$$
$$\dot{b} = Bb + F_2(a,b,y,\epsilon)$$
$$\dot{y} = \epsilon g(a,b,y,\epsilon)$$

$\mathrm{Spec}(A) \subset \{\mathrm{Re}\lambda > 0\}, \ \mathrm{Spec}(B) \subset \{\mathrm{Re}\lambda < 0\}$

$F_1, F_2 = o(|a|,|b|)$

$\to$

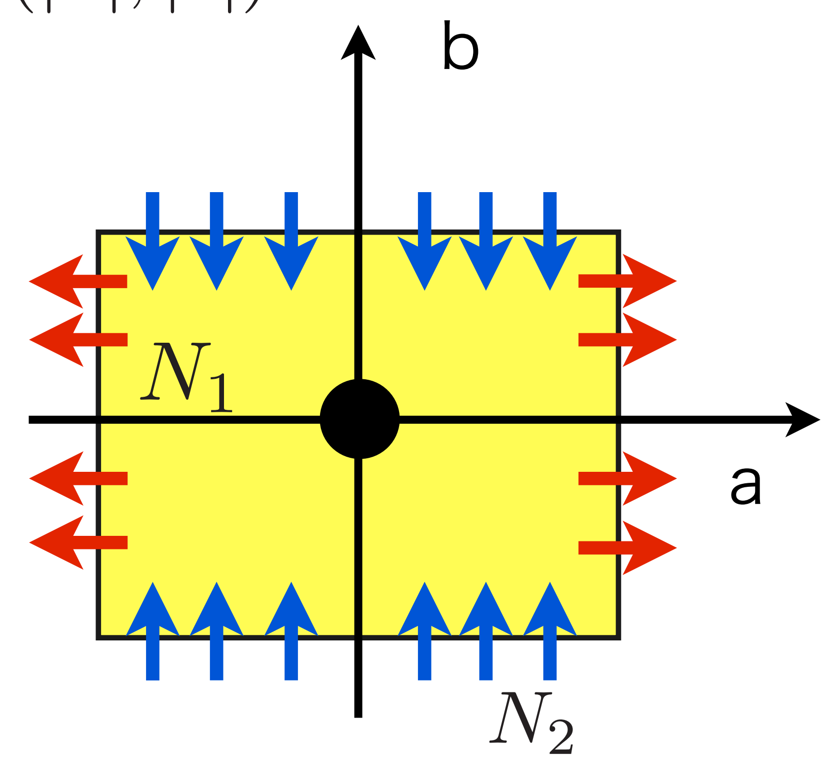
Construct a **fast-saddle-type block** $N \times K$

$K \subset \mathbb{R}^l$: cpt., convex

$N = N_1 \times N_2 \subset \mathbb{R}^n$: cpt., convex s.t.

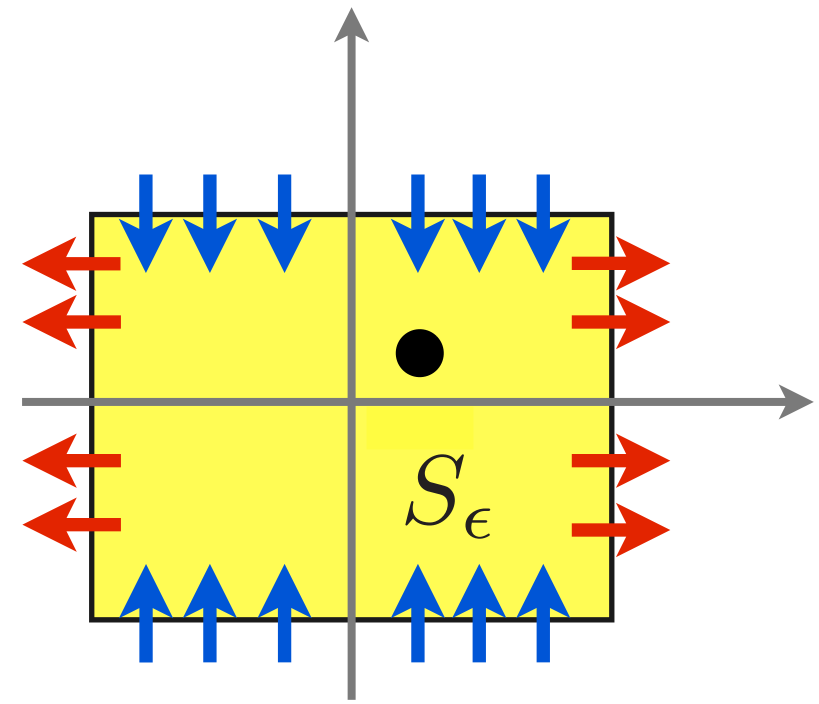
$f(x,y,\epsilon)\cdot \nu_{\partial N_1} > 0$ on $\partial N_1 \times N_2 \times K \times [0,\epsilon_0]$,

$f(x,y,\epsilon)\cdot \nu_{\partial N_2} < 0$ on $N_1 \times \partial N_2 \times K \times [0,\epsilon_0]$



Slow manifold : topological approach

Fast-saddle-type block :
For all y in K, there is a point on slow manifolds **inside the block**.



Proof : Nontriviality of **the topological mapping degree** or **the homological Conley index** (or No Retract Theorem)

Next task :

Dynamics inside blocks

Slow manifold : topological approach

$N \times K$: a fast-saddle-type block

$$\dot{a} = Aa + F_1(a,b,y,\epsilon)$$
$$\dot{b} = Bb + F_2(a,b,y,\epsilon)$$
$$\dot{y} = \epsilon g(a,b,y,\epsilon)$$

Thm [M. cf. Jones (1995) Theorem 4]

In addition, define the following numbers :

$$\sigma^s_{\mathbb{A}_1} : \mathbb{A}_1(z) = \left(\frac{\partial F_1}{\partial a}(z)\right),\ \sigma^s_{\mathbb{A}_2} :\ \mathbb{A}_2(z) = \left(\frac{\partial F_1}{\partial b}(z) \quad \frac{\partial F_1}{\partial y}(z) \quad \frac{\partial F_1}{\partial \eta}(z)\right),$$
$$\sigma^s_{\mathbb{B}_1} : \mathbb{B}_1(z) = \left(\frac{\partial F_2}{\partial a}(z)\right),\ \sigma^s_{\mathbb{B}_2} :\ \mathbb{B}_2(z) = \left(\frac{\partial F_2}{\partial b}(z) \quad \frac{\partial F_2}{\partial y}(z) \quad \frac{\partial F_2}{\partial \eta}(z)\right)$$
$$\sigma^s_{g_1} : g_1(z) = \left(\frac{\partial g}{\partial a}(z)\right),\ \sigma^s_{g_2} :\ g_2(z) = \left(\frac{\partial g}{\partial b}(z) \quad \frac{\partial g}{\partial y}(z) \quad \frac{\partial g}{\partial \eta}(z)\right)$$

Jacobian matrix

's maximal singular values.

Also, assume **the unstable cone condition**

$$\inf \mathrm{Spec}(A) - \left(\sup \sigma^s_{\mathbb{A}_1} + \sup \sigma^s_{\mathbb{A}_2}\right) > 0,$$
$$\inf \mathrm{Spec}(A) + \inf |\mathrm{Spec}(B)|$$
$$- \left\{\sup \sigma^s_{\mathbb{A}_1} + \sup \sigma^s_{\mathbb{A}_2} + \sup \sigma^s_{\mathbb{B}_1} + \sup \sigma^s_{\mathbb{B}_2} + \epsilon_0 \left(\sup \sigma^s_{g_1} + \sup \sigma^s_{g_2}\right)\right\} > 0,$$

in $N \times K \times [0, \epsilon_0]$.

Then there is a smooth function $a = h(b,y,\epsilon)$ such that

$$W^s(S_\epsilon) = \{(a,b,y) \in N \times K \mid a = h(b,y,\epsilon)\},\ \forall \epsilon \in [0,\epsilon_0].$$

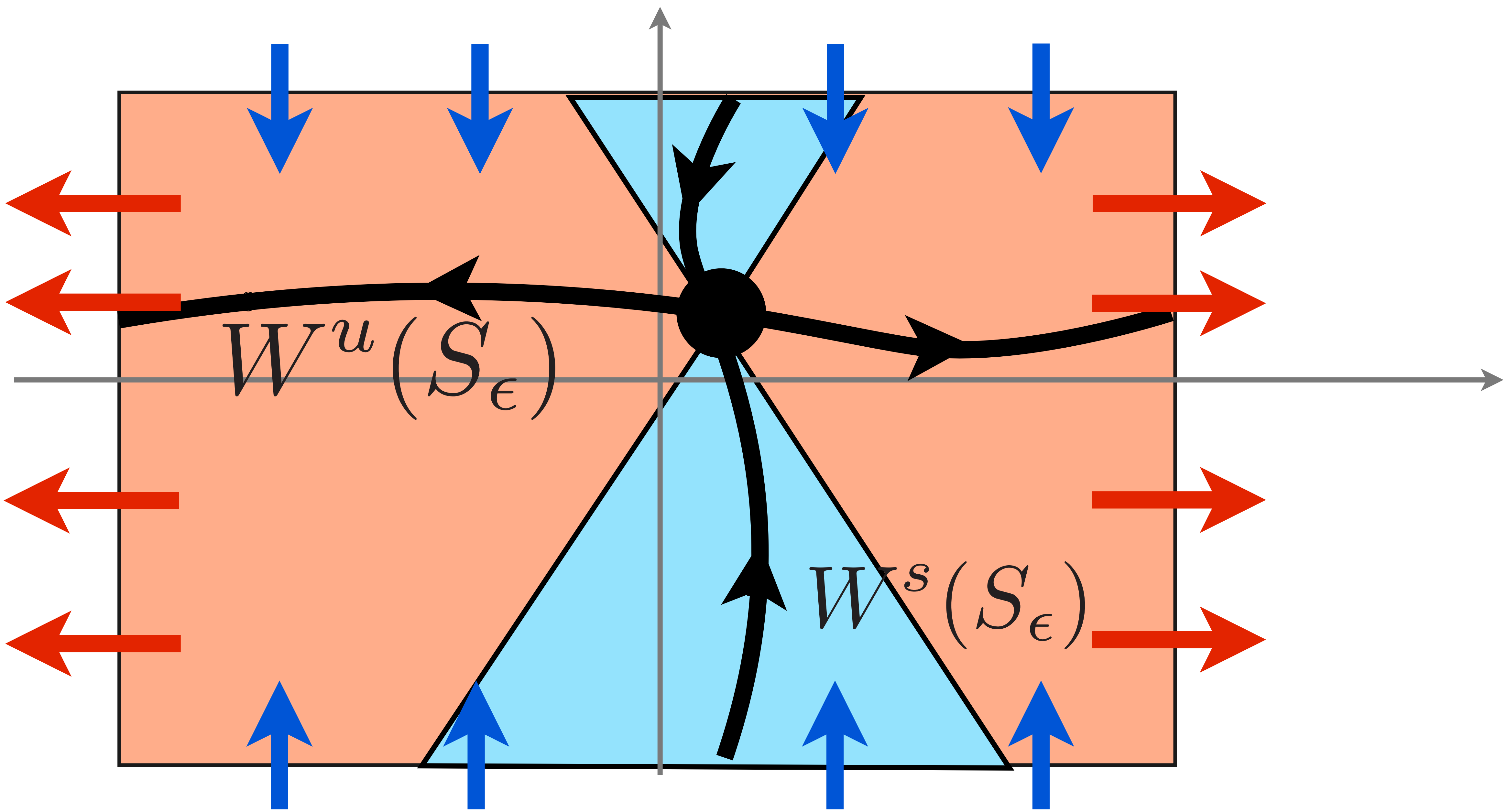
The similar result to $W^u(S_\epsilon)$ also holds.

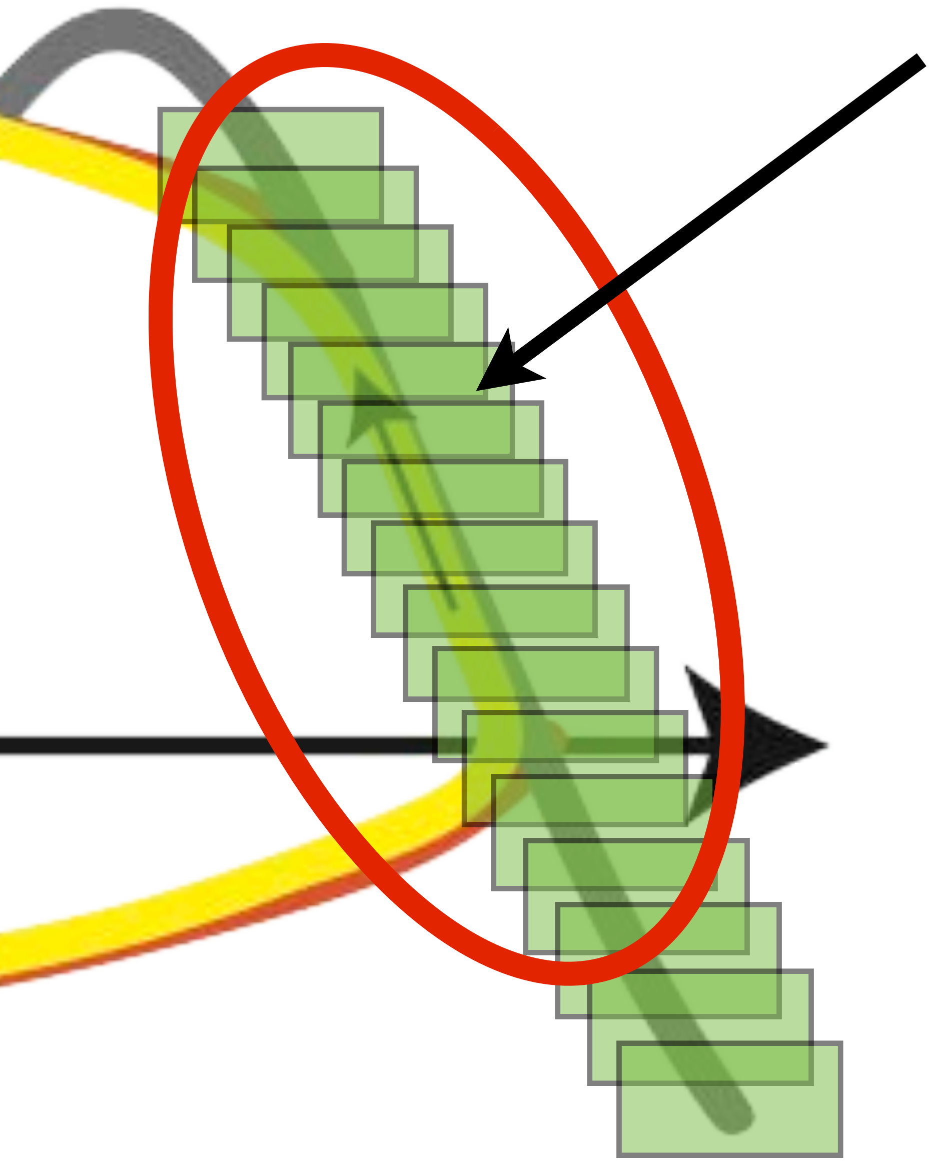
Slow manifold : topological approach

Cone conditions :

The inequality makes a cone below.

(description of (un)stable manifolds : asymptotic behavior)





Our desire :

“trajectories near saddle-type slow manifolds”

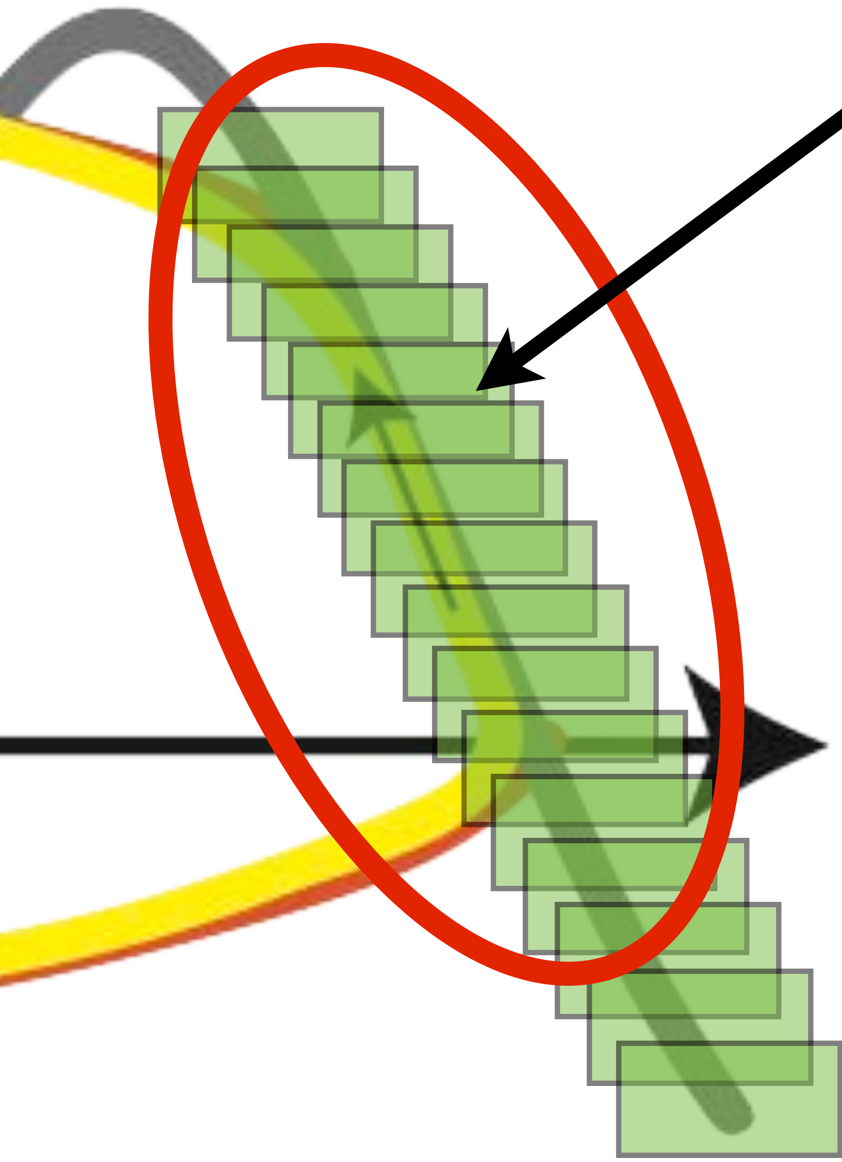
Assumption :
Slow manifold is validated by fast-saddle-type blocks.
(iteration of the previous procedure.)

Restriction : $l = 1$

$$\dot{x} = f(x, y, \epsilon)$$

$$\dot{y} = \epsilon g(x, y, \epsilon), \quad 0 \leq \epsilon \ll 1$$

$x \in \mathbb{R}^n$: fast, $y \in \mathbb{R}^l$: slow, $t \in \mathbb{R}$: time



Our desire :

“trajectories near saddle-type slow manifolds”

Solve solutions directly ?

NO !!!

2. Covering-Exchange

Covering relations

Definition [h-sets, Zgliczynski-Gidea (2002)]

An h-set consists of the following :

$N \subset \mathbb{R}^n$: A compact set

$u(N), s(N) \in \mathbb{Z}_{\geq 0}$ s.t. $u(N) + s(N) = n$

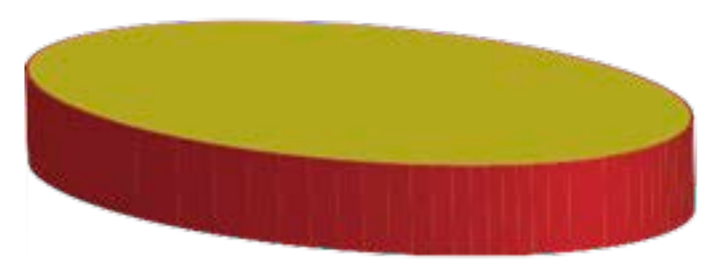
$c_N : \mathbb{R}^n \to \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$: homeomorphism s.t.

$$c_N(N) = \overline{B_{u(N)}} \times \overline{B_{s(N)}}.$$

$u(N)$ -dimensional ball centered at 0, radius 1 $\square$

$$N_c := \overline{B_{u(N)}} \times \overline{B_{s(N)}},$$
$$N_c^- := \partial\overline{B_{u(N)}} \times \overline{B_{s(N)}},$$
$$N_c^+ := \overline{B_{u(N)}} \times \partial\overline{B_{s(N)}},$$
$$N^- := c_N^{-1}(N_c^-), \quad N^+ := c_N^{-1}(N_c^+).$$

Ex : u(N)=1, s(N)=2



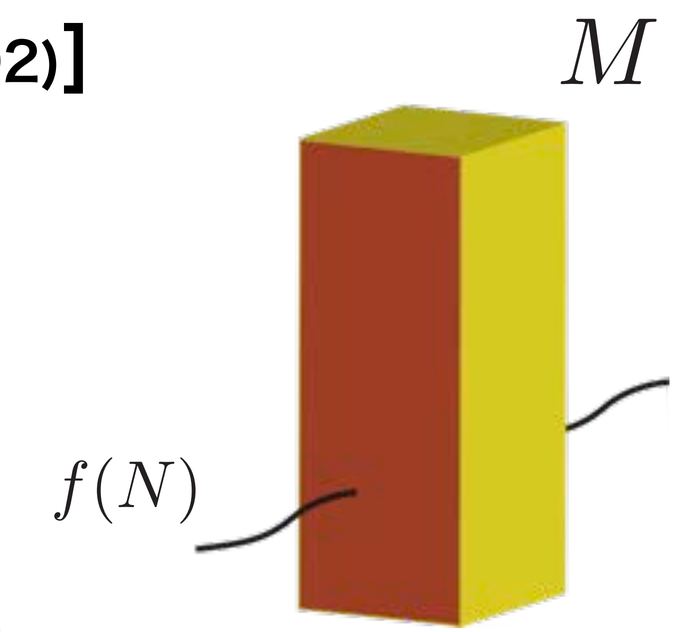
Ex : u(N)=2, s(N)=1

Covering relations

Definition [Covering Relation, Zgliczynski-Gidea (2002)]

N, M : h-sets, $f : N \to \mathbb{R}^{\dim M}$: continuous.

Define $N \overset{f}{\Longrightarrow} M$ as follows :



Ex. : u=1

1. There is a homotopy $h : [0,1] \times N_c \to \mathbb{R}^{\dim M}$ such that

$$h_0 = f_c, \qquad f_c := c_M \circ f \circ c_N^{-1},$$

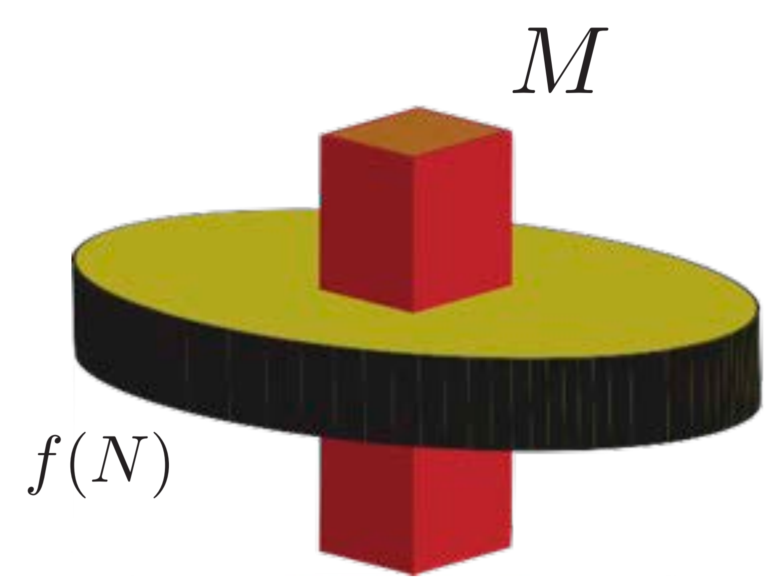
$$h([0,1], N_c^-) \cap M_c = \emptyset,$$

$$h([0,1], N_c) \cap M_c^+ = \emptyset,$$

2. There is a linear map $A : \mathbb{R}^u \to \mathbb{R}^u$ such that

$$h_1(p,q) = (A(p), 0),$$

$$A(\partial B_u(0,1)) \subset \mathbb{R}^u \setminus \overline{B_u}(0,1)$$



Ex. : u=2

Covering relations

Theorem. **[Zgliczynski-Gidea (2002), Wilczak (2006) etc.]**

$\{M_k\}_{k=1}^n$: a sequence of h-sets, $u(M_1) = u(M_2) = \cdots = u(M_k)$

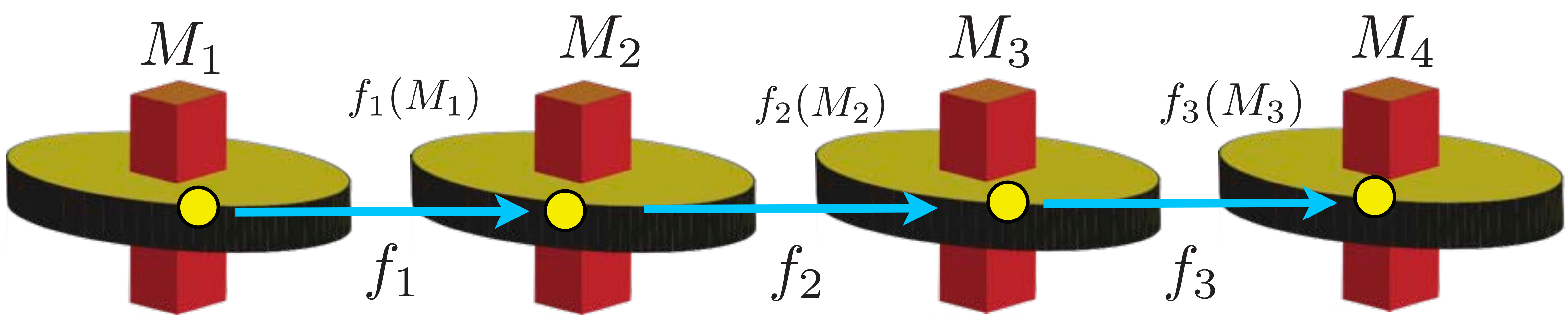
$f_k : M_k \to \mathbb{R}^{\dim M_{k+1}}$: $\mathrm{continuous}$. Assume that

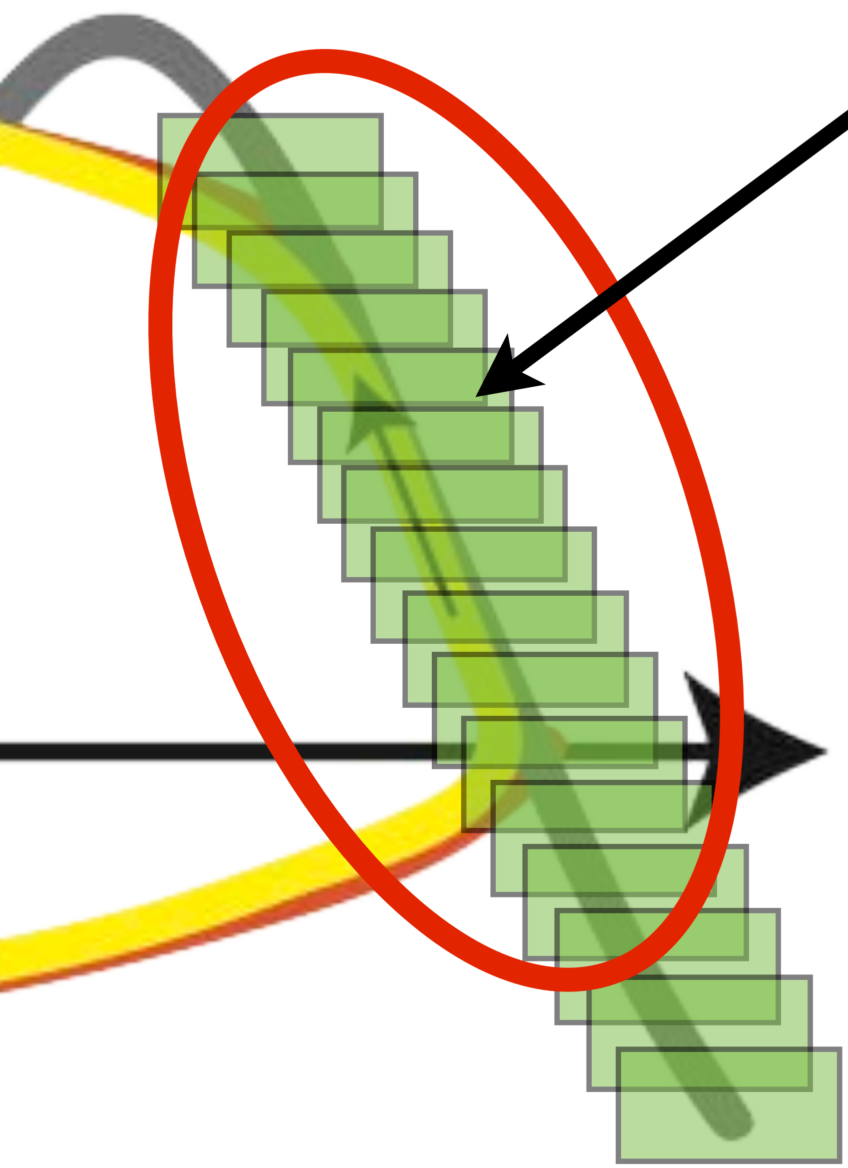
$$M_1 \overset{f_1}{\Longrightarrow} M_2 \overset{f_2}{\Longrightarrow} \cdots \overset{f_{k-1}}{\Longrightarrow} M_k$$

Then

$$\exists x \in M_1 \text{ s.t. } f_i \circ \cdots \circ f_1(x) \in \mathrm{int} M_{i+1}, \quad i = 1, \cdots, k-1.$$

$\square$





Our desire :

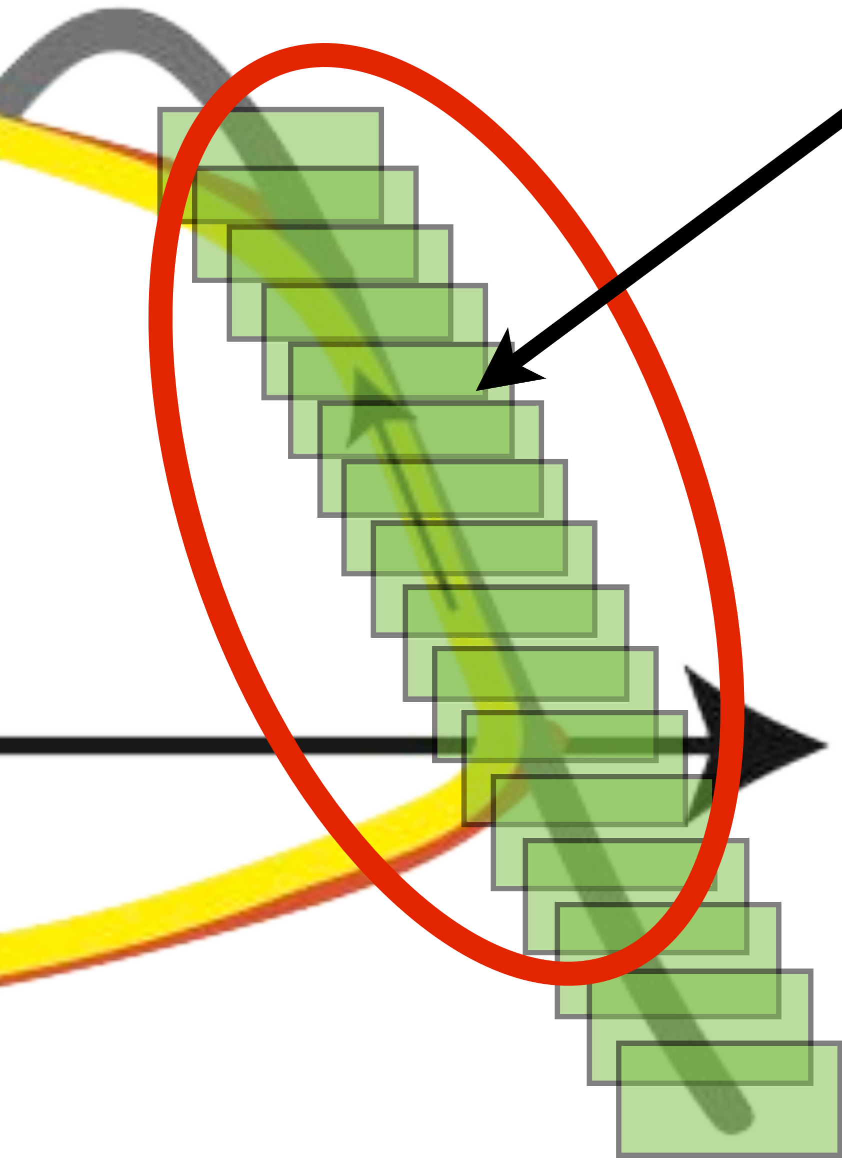
“trajectories near saddle-type slow manifolds”

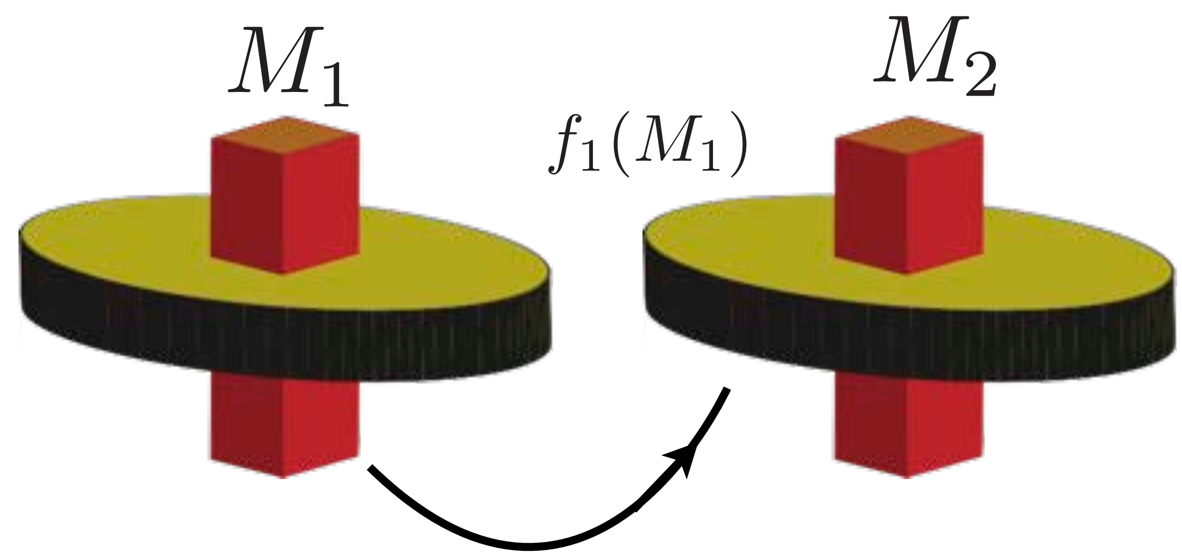
“sequence of covering relations near saddle-type slow manifolds”

$$M_1 \overset{f_1}{\Longrightarrow} M_2 \overset{f_2}{\Longrightarrow} \cdots \overset{f_{k-1}}{\Longrightarrow} M_k$$

Topological interpretation of problems !

“sequence of covering relations near saddle-type slow manifolds”



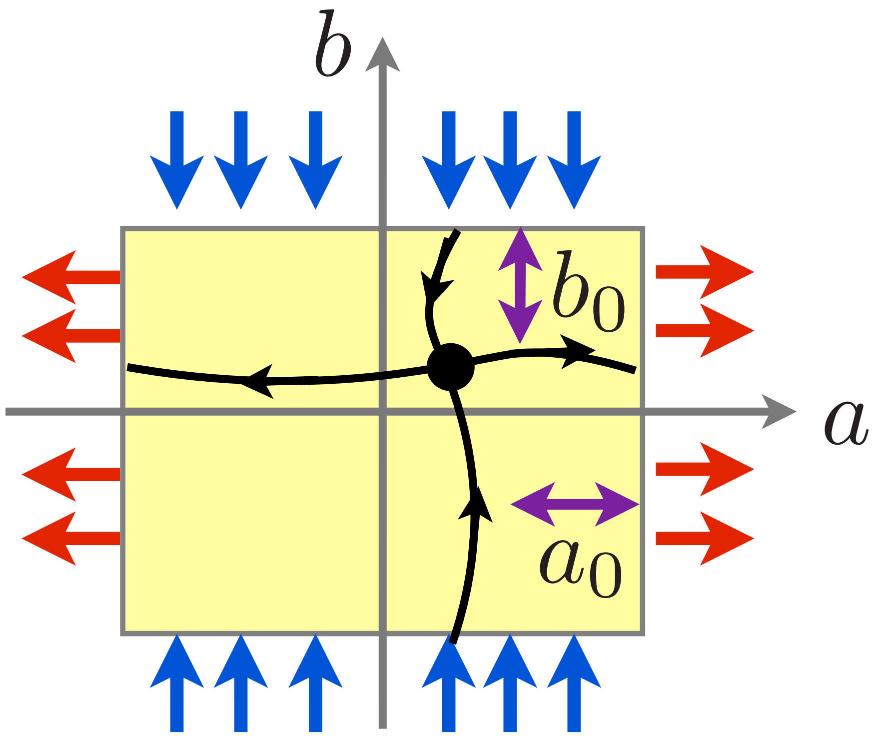


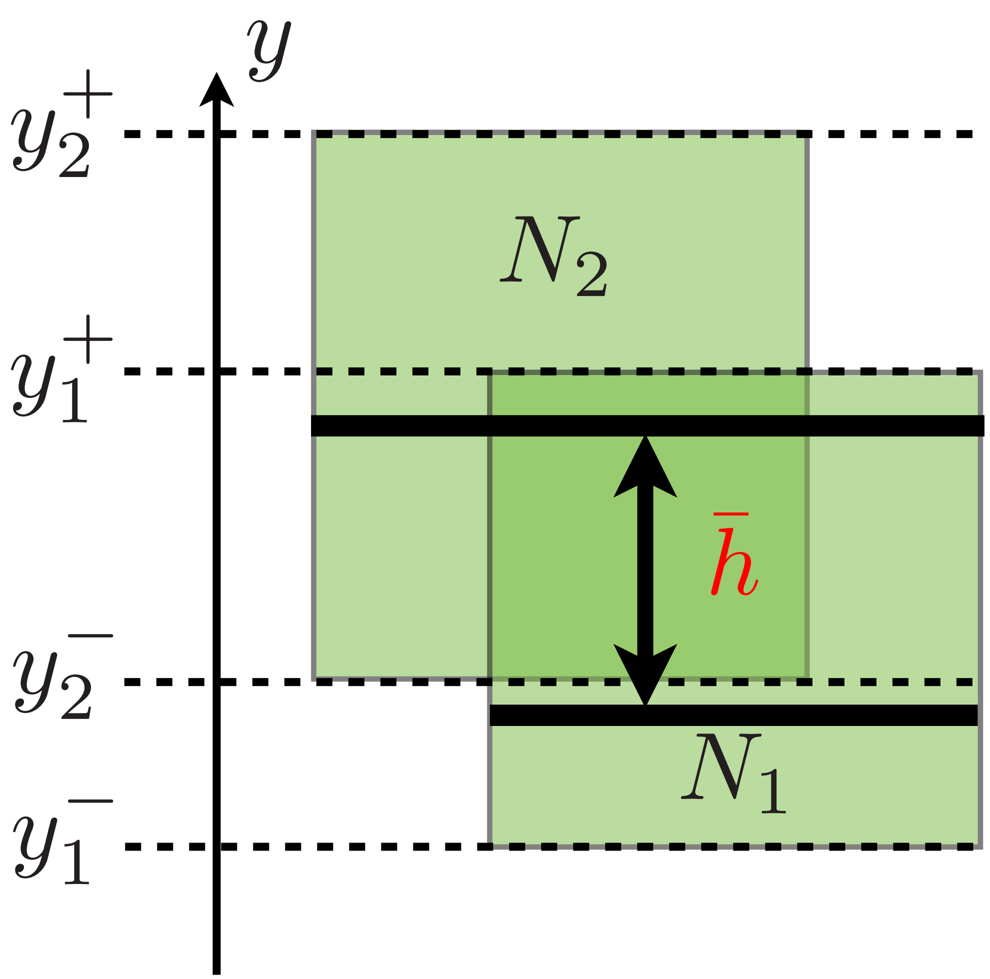
Directly compute covering relations ?
So ridiculous !

near “slow manifolds” …

- Speed of trajectories is **very slow !**
- **Need control of instability.**

Slow shadowing



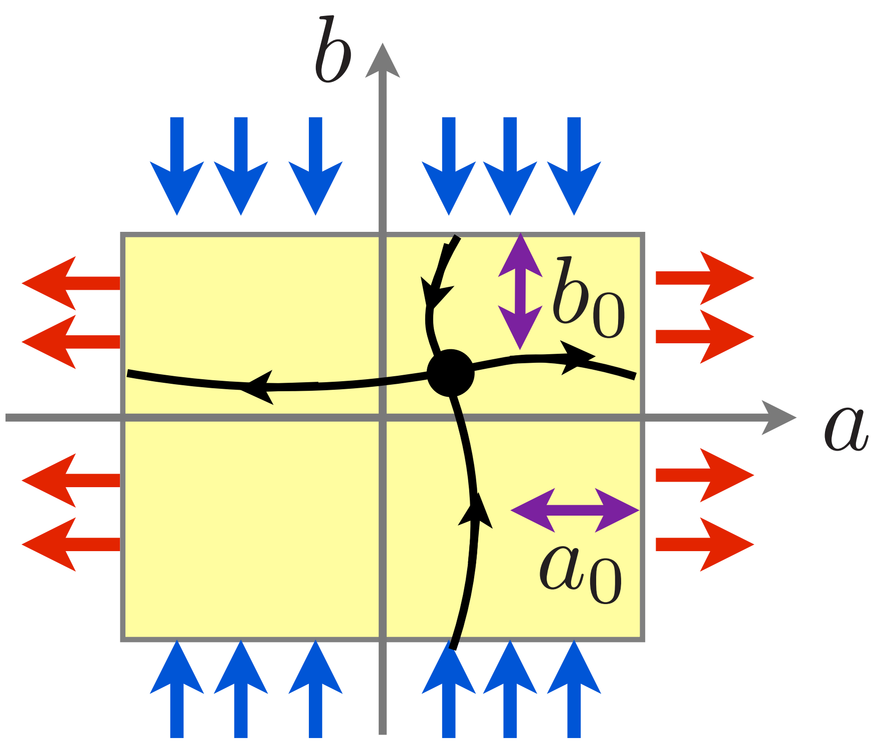


Dynamics around slow manifolds

$$\begin{cases} a' = Aa + F_1(x, y, \epsilon) \\ b' = Bb + F_2(x, y, \epsilon) \\ y' = \epsilon g(x, y, \epsilon) \end{cases}$$

- N_1, N_2 : **Fast-saddle-type blocks w. cones**
- $u(N_1) = u(N_2) = u = 1.$
- $\mathrm{dist}(\partial(\pi_a N_k), \pi_a S_\epsilon) \geq a_0 > 0,$
  $\mathrm{dist}(\partial(\pi_b N_k), \pi_b S_\epsilon) \geq b_0 > 0, \; k = 1, 2.$
- S_ϵ : slow manifold in $N_1 \cup N_2$
- $\bar{a}_k = \mathrm{diam}(\pi_a(N_k)), \; \bar{b}_k = \mathrm{diam}(\pi_b(N_k))$
- $g(x, y, \epsilon) > 0,$
  $\pi_y(N_k) = [y_k^-, y_k^+], \; y_k^- < y_k^+ \; (k = 1, 2),$
  $y_2^- \in (y_1^-, y_1^+], \quad y_1^+ \in [y_2^-, y_2^+).$

Slow shadowing



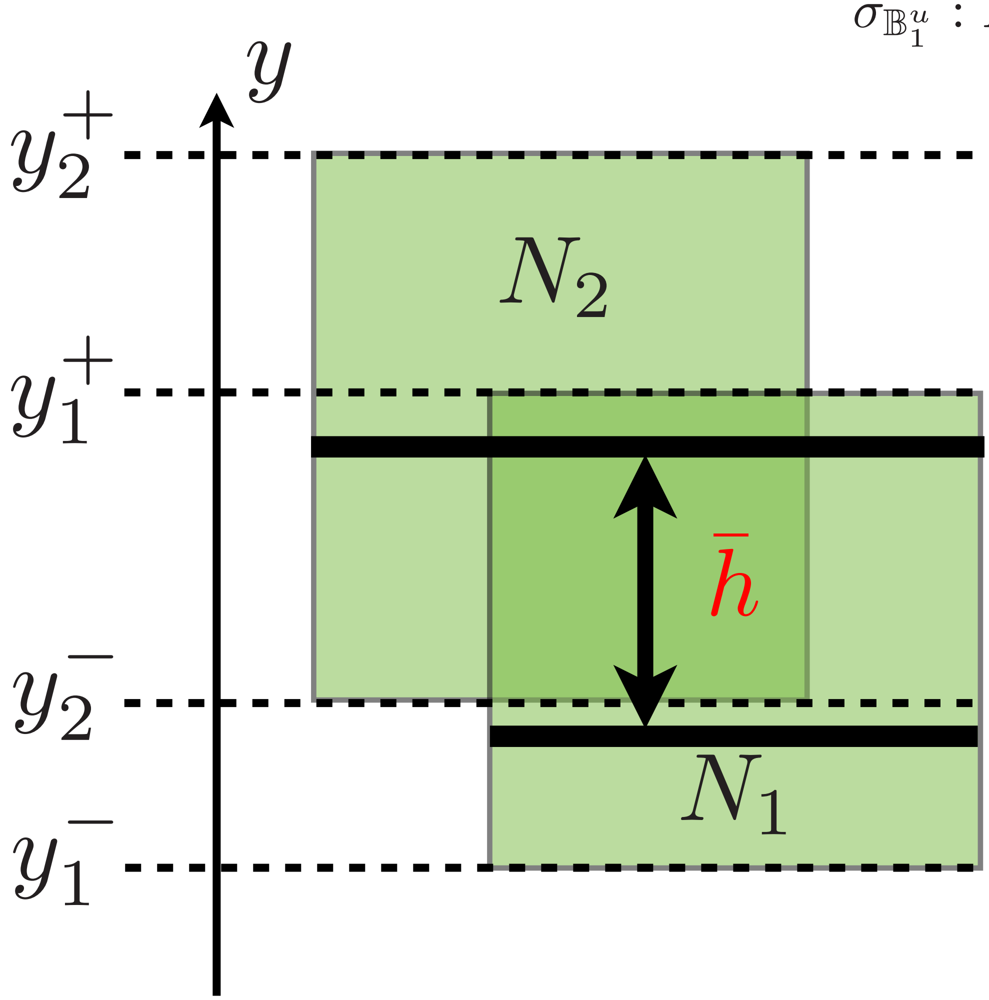
$$\lambda_k := \alpha_k - \left(\sup_{N_k \times [0,\epsilon_0]} \sigma_{\mathbb{A}_1^u} + \sup_{N_k \times [0,\epsilon_0]} \sigma_{\mathbb{A}_2^u} \right),$$

$$\mu_k := \beta_k + \left(\sup_{N_k \times [0,\epsilon_0]} \sigma_{\mathbb{B}_1^s} + \sup_{N_k \times [0,\epsilon_0]} \sigma_{\mathbb{B}_2^s} \right).$$

$$\sigma_{\mathbb{A}_1^u} : \mathbb{A}_1^u(z) = \left(\frac{\partial F_1}{\partial a}(z) \right), \quad \sigma_{\mathbb{A}_2^u} : \mathbb{A}_2^u(z) = \left(\frac{\partial F_1}{\partial b}(z) \quad \frac{\partial F_1}{\partial y}(z) \quad \frac{\partial F_1}{\partial \eta}(z) \right),$$

$$\sigma_{\mathbb{B}_1^u} : \mathbb{B}_1^u(z) = \left(\frac{\partial F_2}{\partial a}(z) \right), \quad \sigma_{\mathbb{B}_2^u} : \mathbb{B}_2^u(z) = \left(\frac{\partial F_2}{\partial b}(z) \quad \frac{\partial F_2}{\partial y}(z) \quad \frac{\partial F_2}{\partial \eta}(z) \right)$$

's maximal singular values



Definition.

We say a pair $\{N_1, N_2\}$ of h-sets is a **slow shadowing pair** if

$$\max\left\{ \frac{1}{\lambda_k} \log\left(\frac{\bar{a}_k}{a_0}\right), \frac{1}{|\mu_k|} \log\left(\frac{\bar{b}_k}{b_0}\right) \right\} < \frac{\bar{h}}{\bar{\epsilon}_k},$$

$k = 1, 2.$

Slow shadowing

Proposition **[slow shadowing, M., arXiv 1507.01462]**

For a slow shadowing pair $\{N_1, N_2\}$, there are h-sets

$$M_1 \subset (N_1)_{\bar y} = N_1 \cap \{y = \bar y\}, \quad M_2 \subset (N_2)_{\bar y + \bar h} = N_2 \cap \{y = \bar y + \bar h\}$$

such that $M_1 \overset{P_\epsilon^{(N_1)_{\leq \bar y + \bar h}}}{\Longrightarrow} M_2.$

$$\max\left\{\frac{1}{\lambda_k}\log\left(\frac{\bar a_k}{a_0}\right), \frac{1}{|\mu_k|}\log\left(\frac{\bar b_k}{b_0}\right)\right\} < \frac{\bar h}{\bar \epsilon_k},$$

$k = 1, 2.$

Slow shadowing

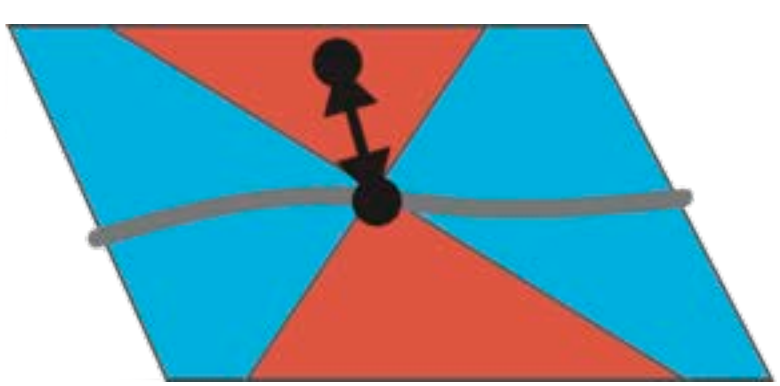
$$\max\left\{\frac{1}{\lambda_k}\log\left(\frac{\bar{a}_k}{a_0}\right), \frac{1}{|\mu_k|}\log\left(\frac{\bar{b}_k}{b_0}\right)\right\} < \frac{\bar{h}}{\bar{\epsilon}_k},$$

$$k = 1, 2.$$

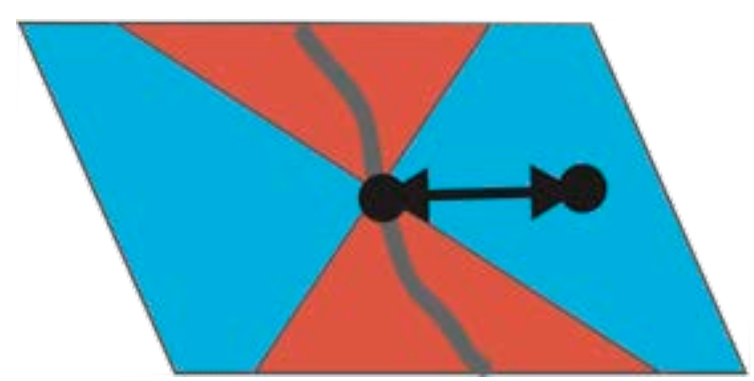
$$|a(t) - a_M(t)| \geq e^{\lambda_k t}|a(0) - a_M(0)|$$

$$|b(-t) - b_M(-t)| \geq e^{-\mu_k t}|b(0) - b_M(0)|$$

$$\bar{\epsilon}_k \geq \epsilon g(x, y, \epsilon)$$

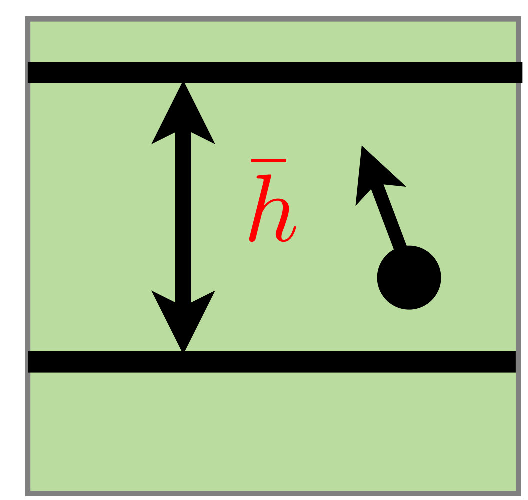


“moving unstable cone” for each point



“moving stable cone” for each point

vs.



speed in slow direction

Slow shadowing

Proposition [slow shadowing, M., arXiv 1507.01462]

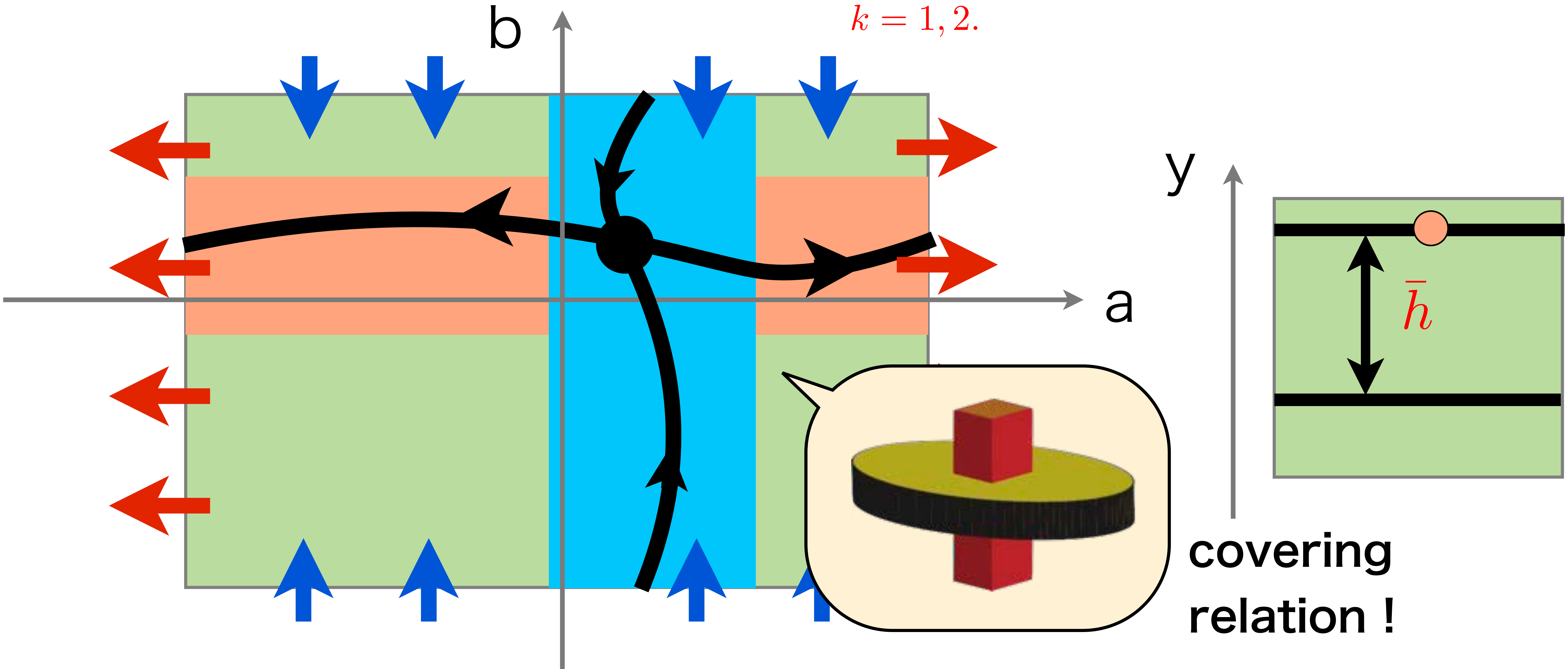
For a slow shadowing pair $\{N_1, N_2\}$, there are h-sets

$$M_1 \subset (N_1)_{\bar y} = N_1 \cap \{y = \bar y\},\quad M_2 \subset (N_2)_{\bar y + \bar h} = N_2 \cap \{y = \bar y + \bar h\}$$

such that $M_1 \overset{P_\epsilon^{(N_1)_{\leq \bar y + \bar h}}}{\Longrightarrow} M_2.$

$$\max \left\{ \frac{1}{\lambda_k} \log\left(\frac{\bar a_k}{a_0}\right), \frac{1}{|\mu_k|} \log\left(\frac{\bar b_k}{b_0}\right) \right\} < \frac{\bar h}{\bar \epsilon_k},$$

$k = 1, 2.$



covering relation !

Slow shadowing

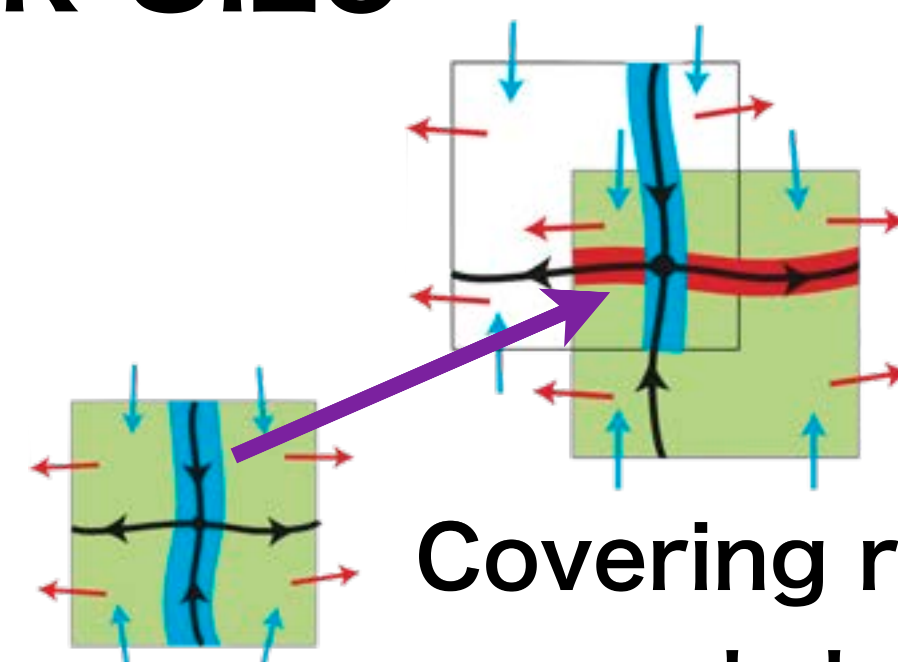
$$\max_{k=1,2.}\left\{\frac{1}{\lambda_k}\log\left(\frac{\bar{a}_k}{a_0}\right),\frac{1}{|\mu_k|}\log\left(\frac{\bar{b}_k}{b_0}\right)\right\}<\frac{\bar{h}}{\bar{\epsilon}_k},$$

Eigenvalues

Block Size

Bound of speed

Very simple calculations



Covering relation around slow manifolds !

Covering-Exchange

Proposition [Covering-Exchange : drop, M.]

For a slow shadowing pair $\{N_1, N_2\}$, assume that $N \overset{\varphi_\epsilon(T,\cdot)}{\Longrightarrow} (N_1)_{\leq \bar{y}}$.

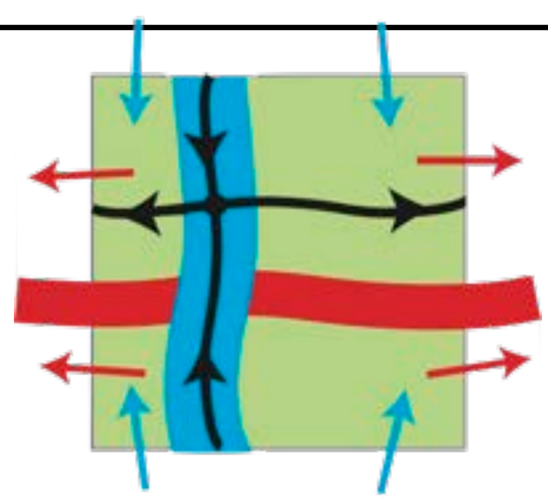
for some h-set N. Then there are two h-sets $M_1 \subset (N_1)_{\bar{y}},\ M_2 \subset (N_2)_{\bar{y}+\bar{h}}$

such that $N \overset{\varphi_\epsilon(T,\cdot)}{\Longrightarrow} M_1 \overset{P_\epsilon^{(N_1)_{\leq \bar{y}+\bar{h}}}}{\Longrightarrow} M_2$.

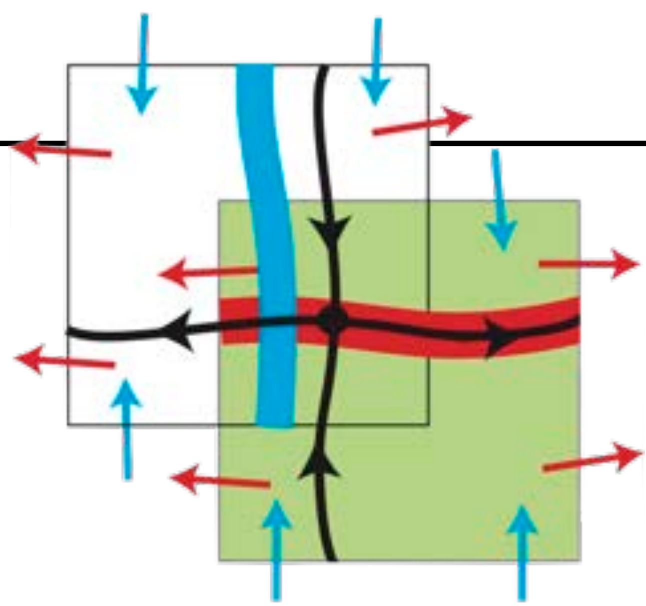
Proposition [Covering-Exchange : jump, M.]

For a slow shadowing pair $\{N_1, N_2\}$, let $N_2^{\mathrm{exit}} \subset N_2^{f,-}$ be a fast-exit face of N_2 (subset of the boundary with red arrow in Fig.2) and assume $\mathrm{dist}(N_2^{\mathrm{exit}}, \{y = \bar{y}\}) > 2\bar{h}$. Then there are two h-sets $M_1 \subset (N_1)_{\bar{y}},\ M_2 \subset (N_2)_{\bar{y}+\bar{h}}$

such that $M_1 \overset{P_\epsilon^{(N_1)_{\leq \bar{y}+\bar{h}}}}{\Longrightarrow} M_2 \overset{P_\epsilon^{N_2}}{\Longrightarrow} N_2^{\mathrm{exit}}$.

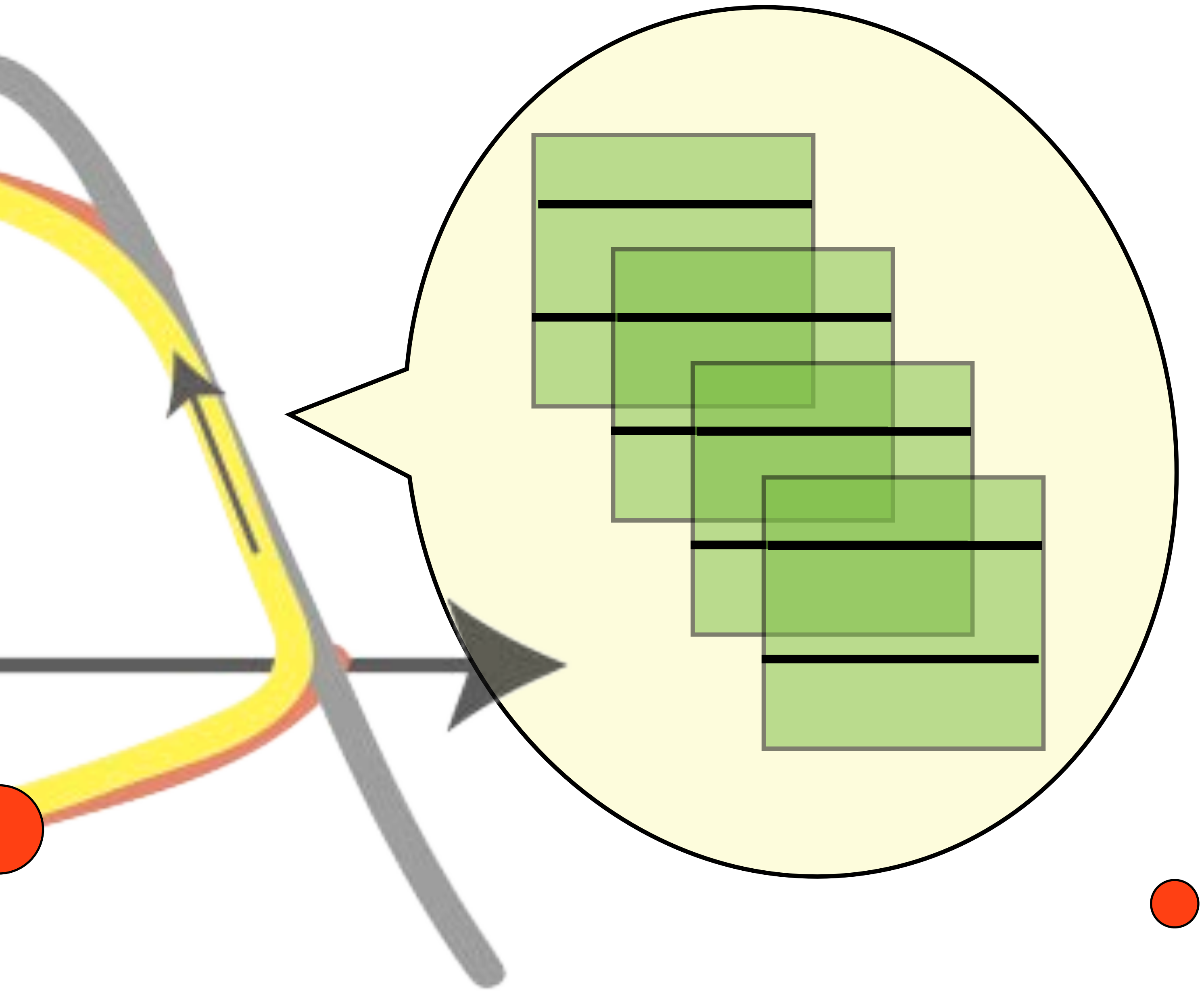


drop

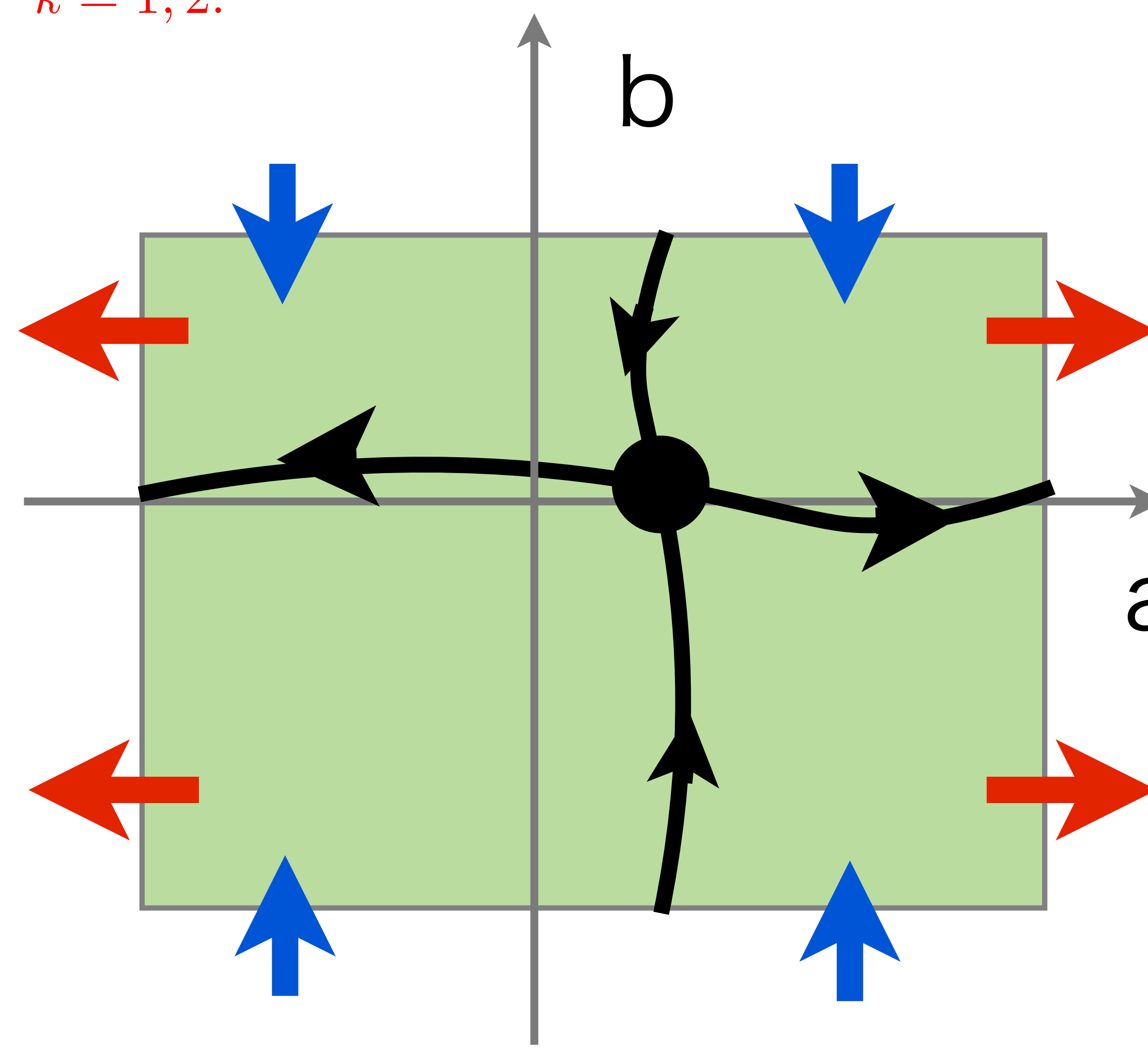


jump

“drop”



$$\max\left\{\frac{1}{\lambda_k}\log\left(\frac{\bar{a}_k}{a_0}\right),\frac{1}{|\mu_k|}\log\left(\frac{\bar{b}_k}{b_0}\right)\right\}<\frac{\bar{h}}{\bar{\epsilon}_k},$$
$k=1,2.$

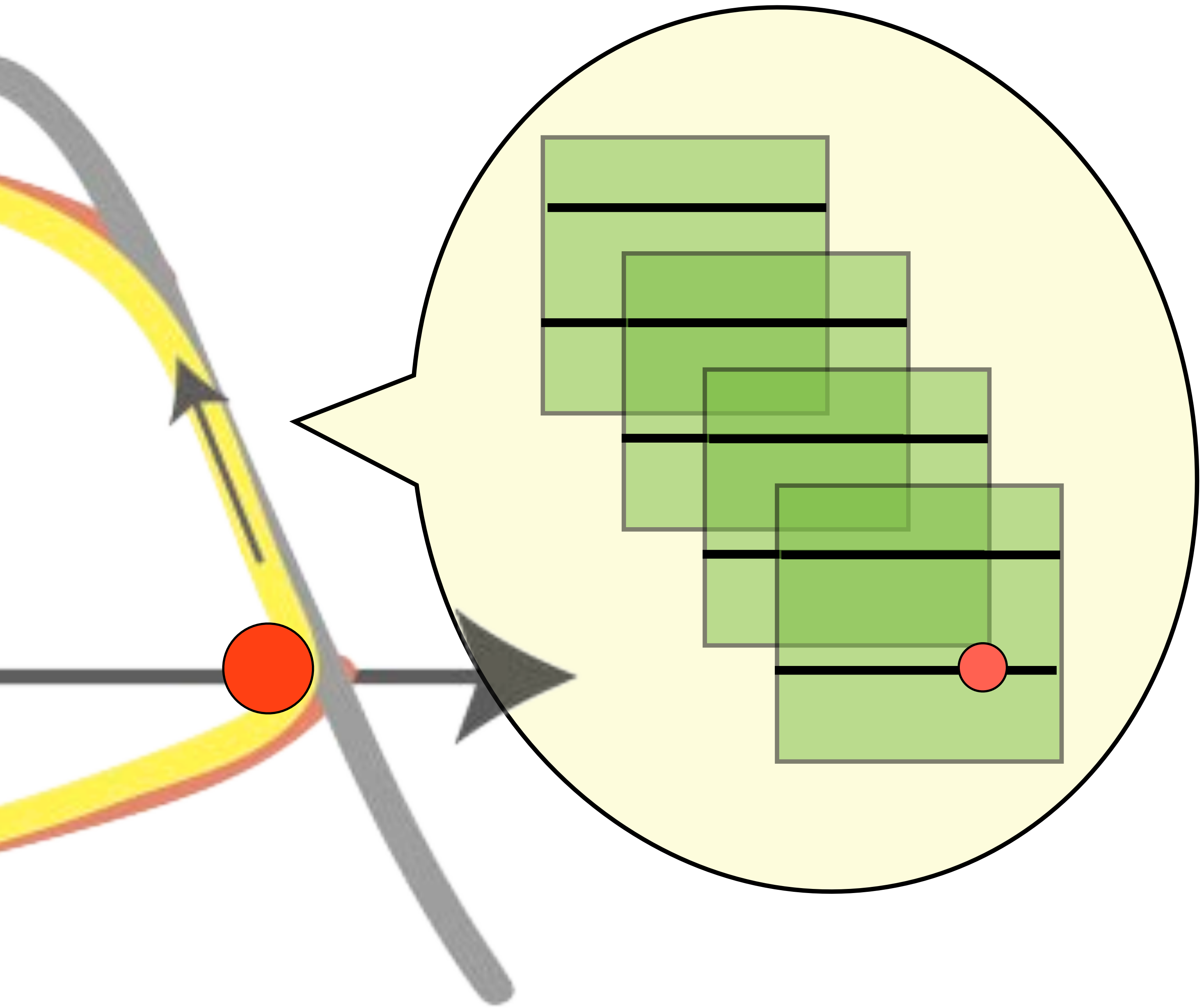


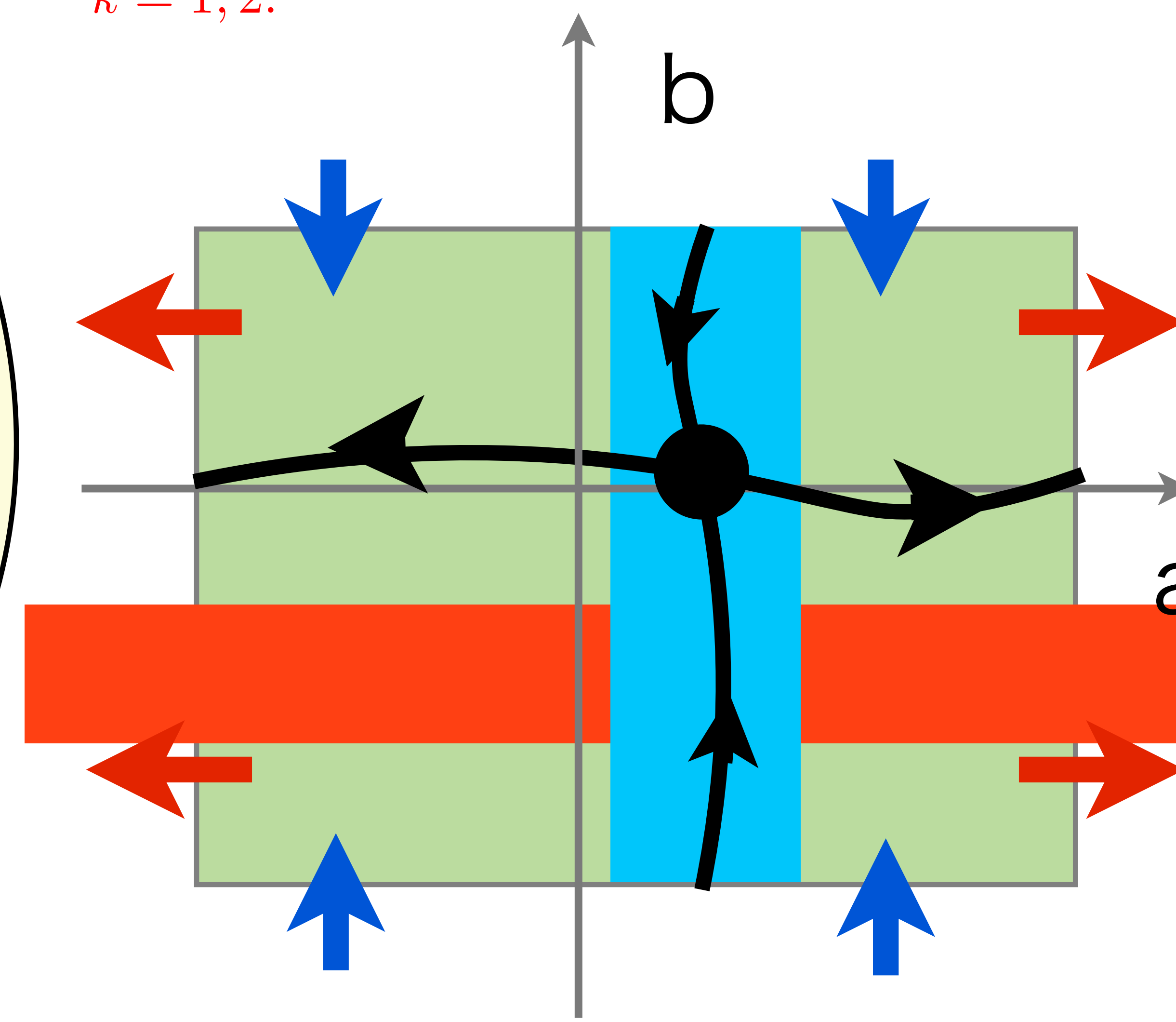
$$\boxed{N} \xRightarrow{\rho_\epsilon(T,\cdot)} M_1 \xRightarrow{P_\epsilon^1} M_2 \xRightarrow{P_\epsilon^2} \cdots \xRightarrow{P_\epsilon^{m-1}} M_m \xRightarrow{P_\epsilon^m} N_{m+1}^{\mathrm{exit}}.$$

“Covering-Exchange”

"drop"

$$\max\left\{\frac{1}{\lambda_k}\log\left(\frac{\bar{a}_k}{a_0}\right), \frac{1}{|\mu_k|}\log\left(\frac{\bar{b}_k}{b_0}\right)\right\} < \frac{\bar{h}}{\bar{\epsilon}_k},$$
$k = 1, 2.$

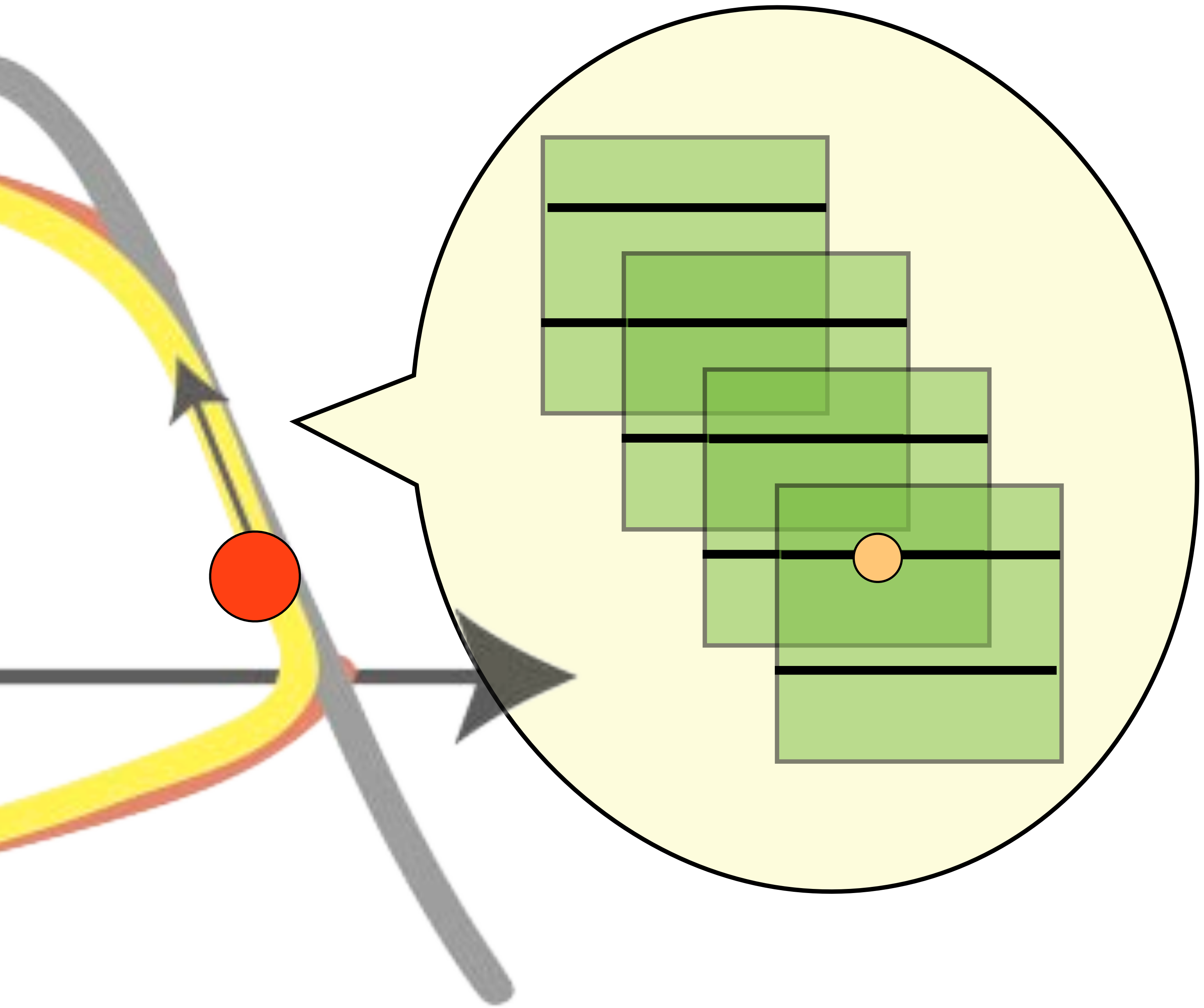




$$\boxed{N \overset{\varphi_\epsilon(T,\cdot)}{\Longrightarrow} M_1} \overset{P_\epsilon^1}{\Longrightarrow} M_2 \overset{P_\epsilon^2}{\Longrightarrow} \cdots \overset{P_\epsilon^{m-1}}{\Longrightarrow} M_m \overset{P_\epsilon^m}{\Longrightarrow} N_{m+1}^{\mathrm{exit}}.$$

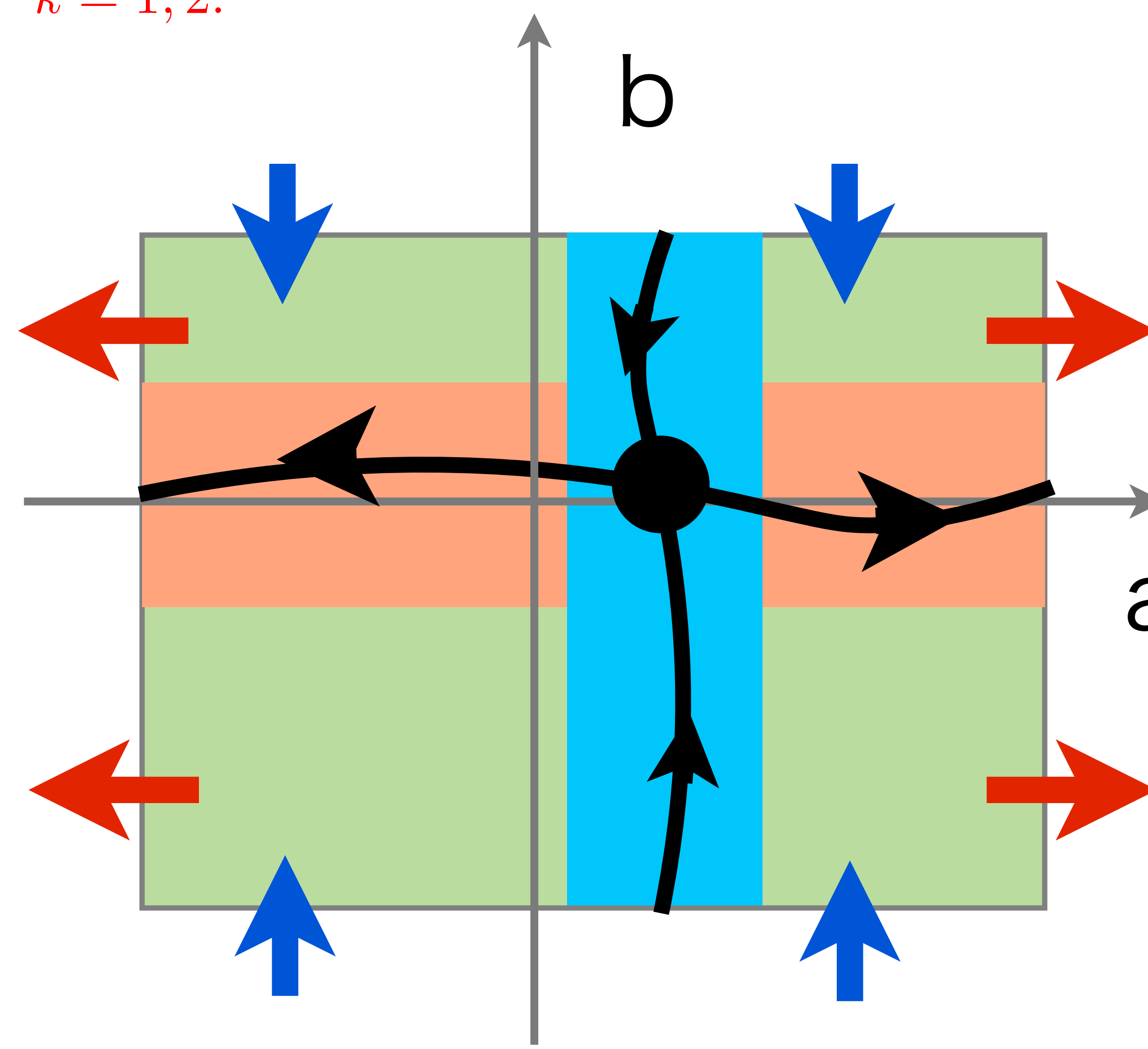
"Covering-Exchange"

“slow shadowing”



$$\max\left\{\frac{1}{\lambda_k}\log\left(\frac{\bar{a}_k}{a_0}\right), \frac{1}{|\mu_k|}\log\left(\frac{\bar{b}_k}{b_0}\right)\right\} < \frac{\bar{h}}{\bar{\epsilon}_k},$$

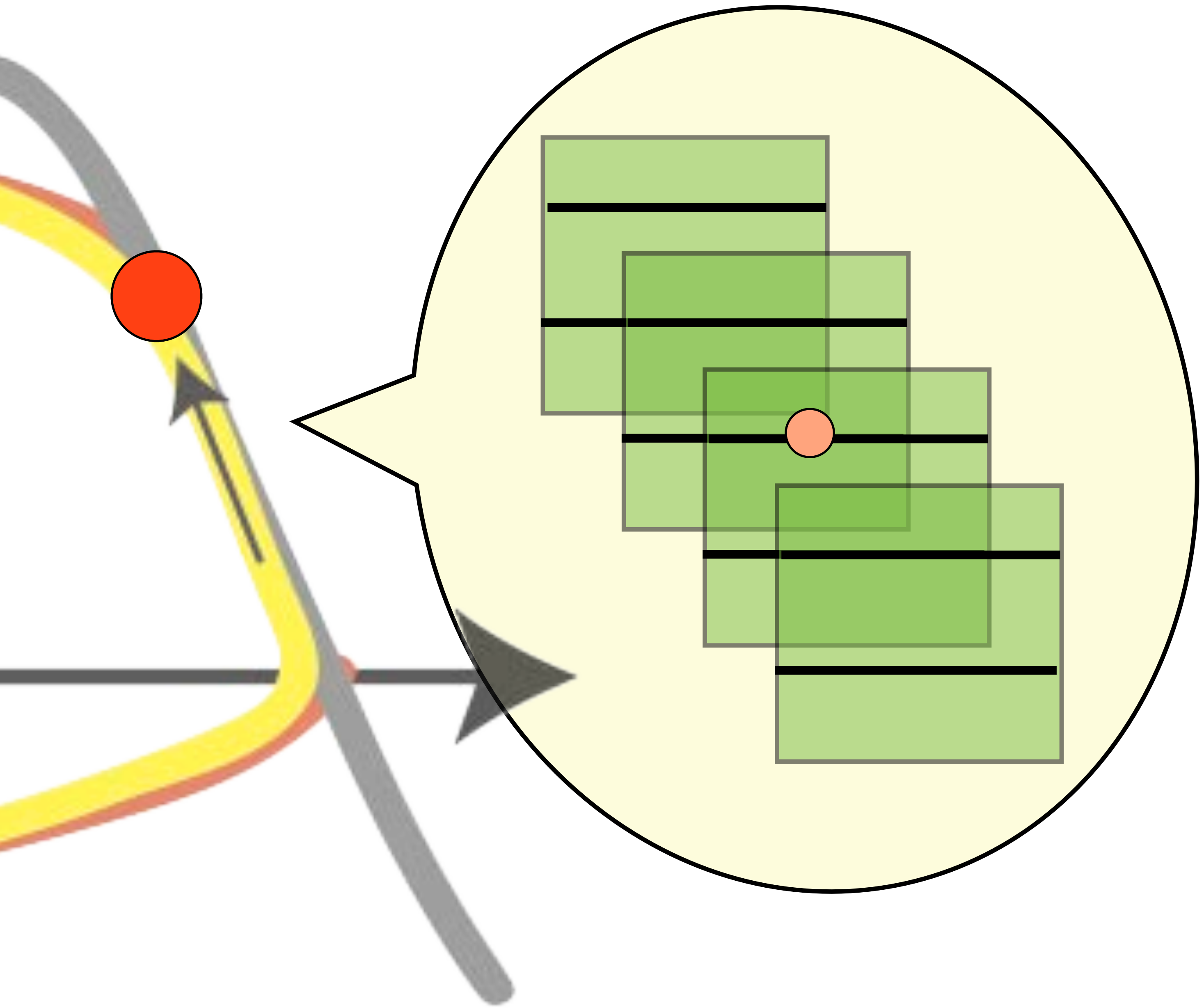
$k = 1, 2.$



$$N \overset{\varphi_\epsilon(T,\cdot)}{\Longrightarrow} \boxed{M_1 \overset{P_\epsilon^1}{\Longrightarrow} M_2} \overset{P_\epsilon^2}{\Longrightarrow} \cdots \overset{P_\epsilon^{m-1}}{\Longrightarrow} M_m \overset{P_\epsilon^m}{\Longrightarrow} N_{m+1}^{\mathrm{exit}}.$$

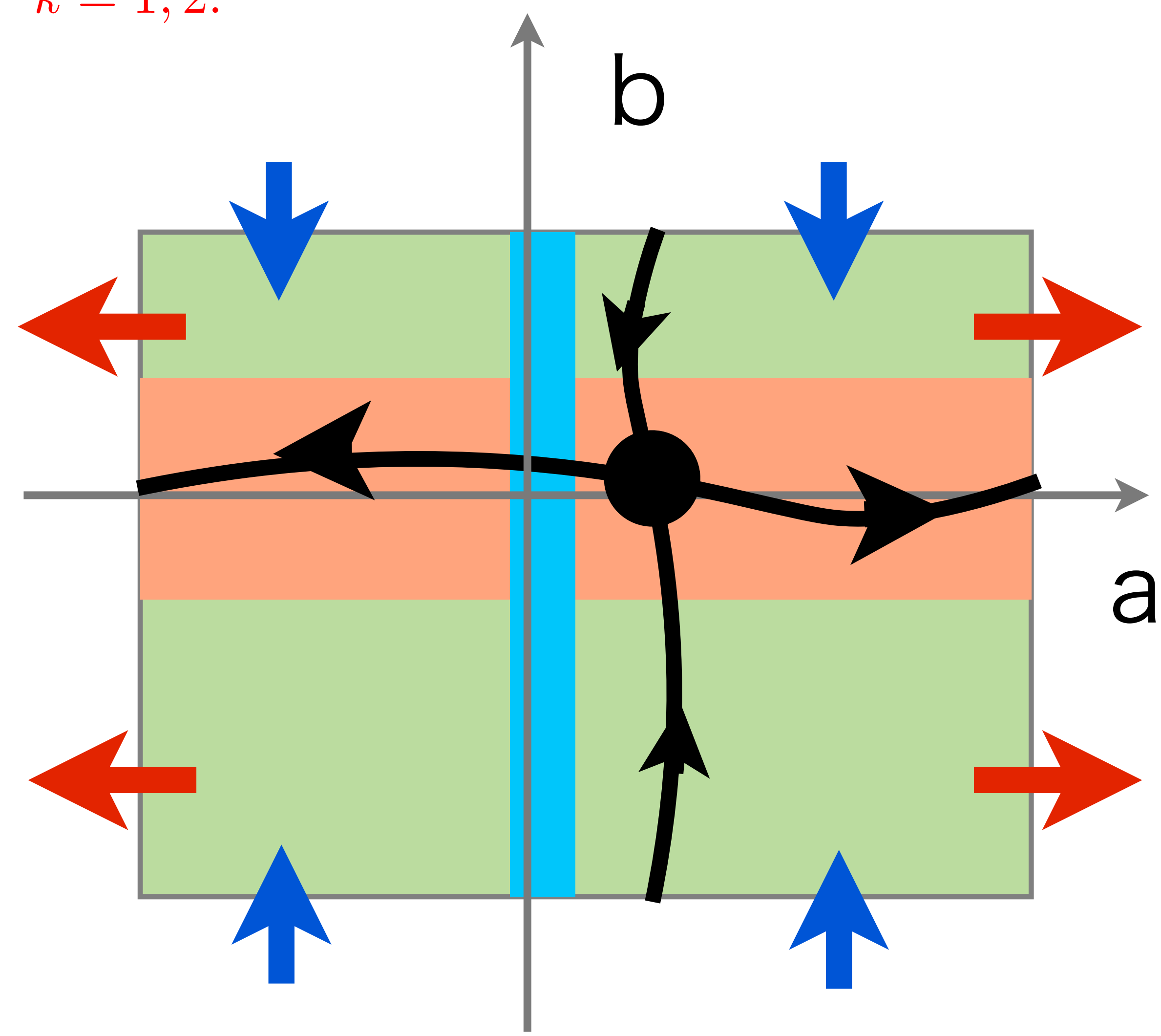
“Covering-Exchange”

“jump”



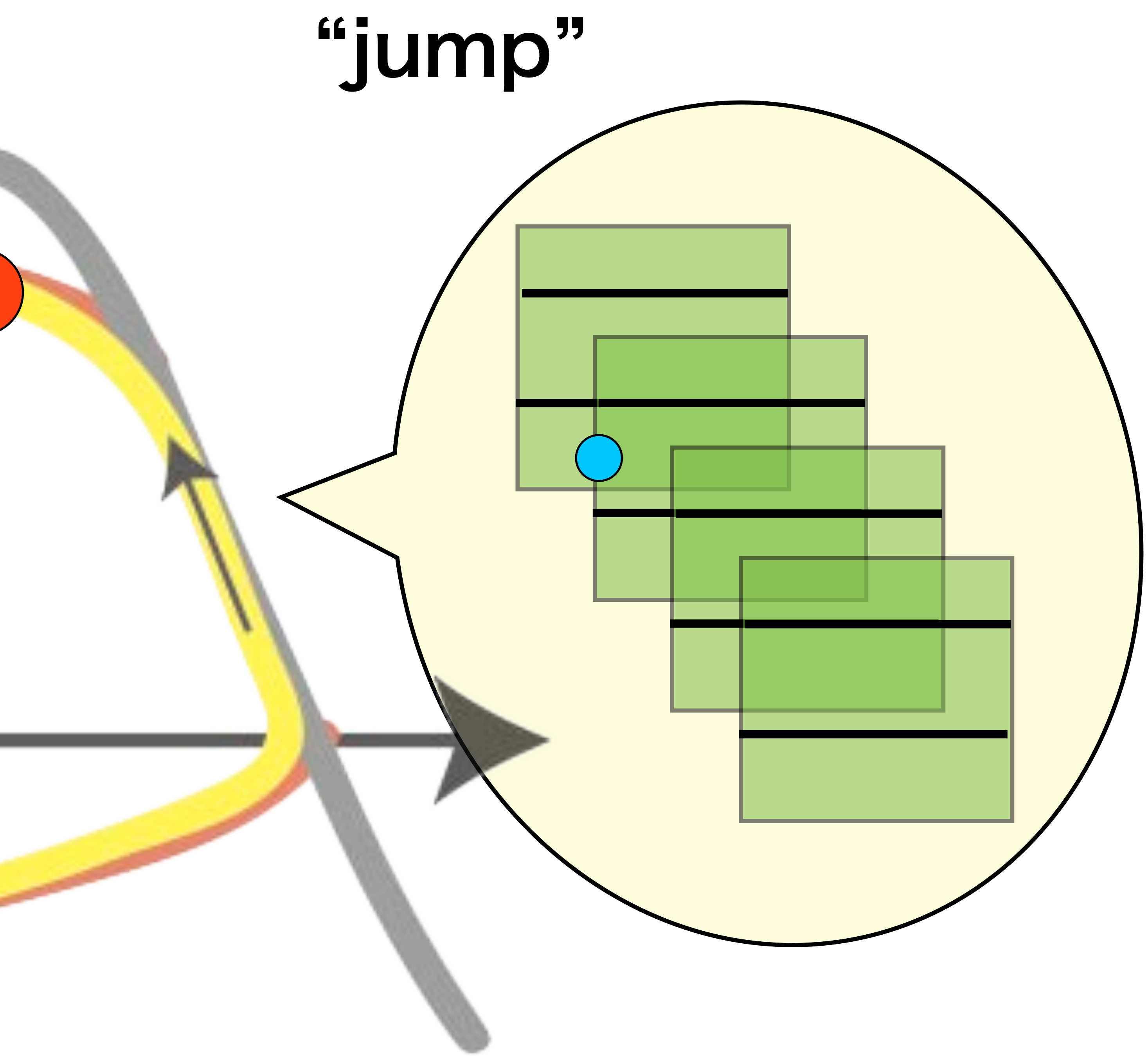
$$\max\left\{\frac{1}{\lambda_k}\log\left(\frac{\bar{a}_k}{a_0}\right), \frac{1}{|\mu_k|}\log\left(\frac{\bar{b}_k}{b_0}\right)\right\} < \frac{\bar{h}}{\bar{\epsilon}_k},$$

$k = 1, 2.$



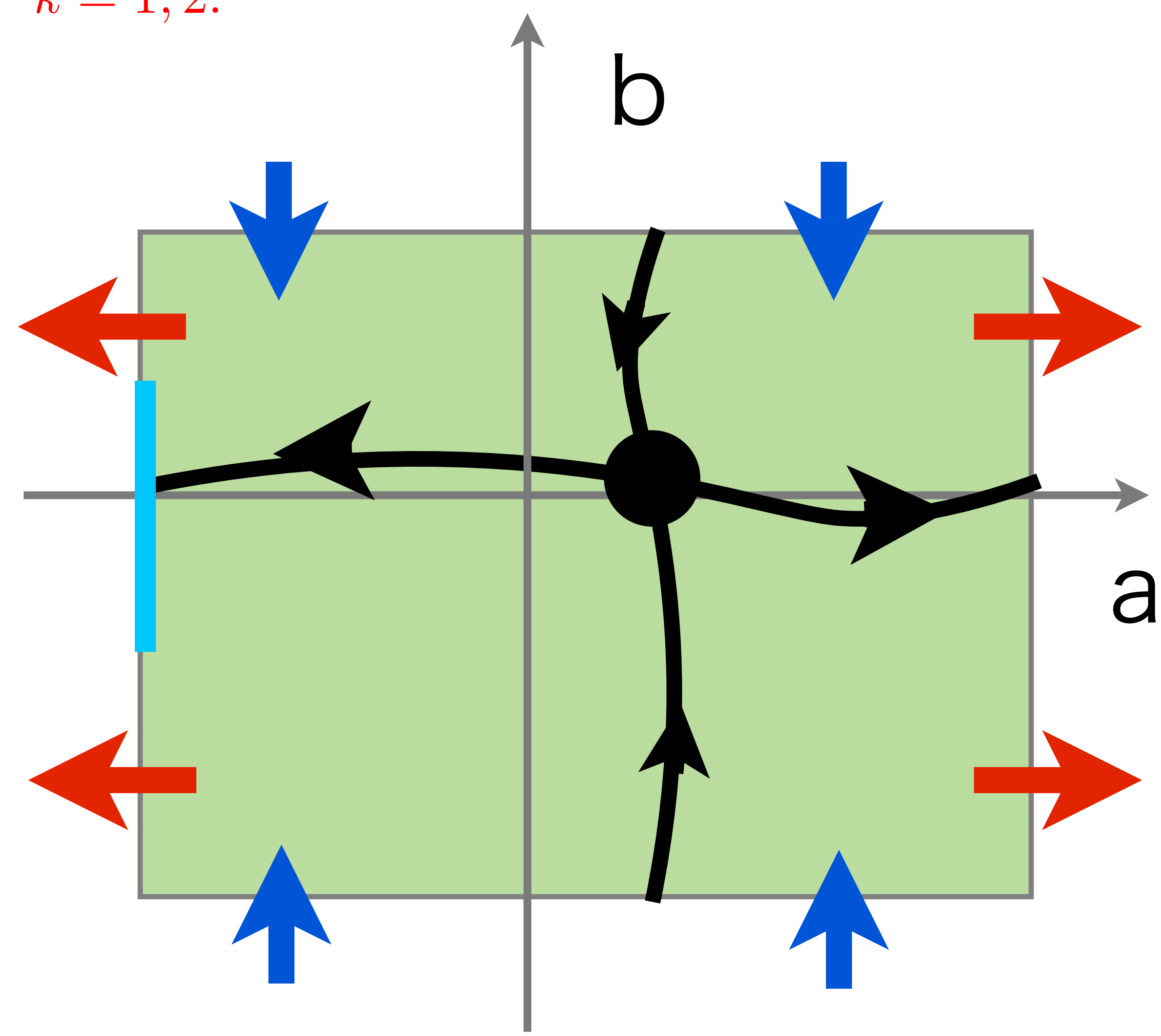
$$N \overset{\varphi_\epsilon(T,\cdot)}{\Longrightarrow} M_1 \overset{P_\epsilon^1}{\Longrightarrow} M_2 \overset{P_\epsilon^2}{\Longrightarrow} \cdots \overset{P_\epsilon^{m-1}}{\Longrightarrow} \boxed{M_m \overset{P_\epsilon^m}{\Longrightarrow} N_{m+1}^{\mathrm{exit}}.}$$

“Covering-Exchange”



$$\max\left\{\frac{1}{\lambda_k}\log\left(\frac{\bar{a}_k}{a_0}\right), \frac{1}{|\mu_k|}\log\left(\frac{\bar{b}_k}{b_0}\right)\right\} < \frac{\bar{h}}{\bar{\epsilon}_k},$$

$k = 1, 2.$



$$N \overset{\varphi_\epsilon(T,\cdot)}{\Longrightarrow} M_1 \overset{P_\epsilon^1}{\Longrightarrow} M_2 \overset{P_\epsilon^2}{\Longrightarrow} \cdots \overset{P_\epsilon^{m-1}}{\Longrightarrow} M_m \overset{P_\epsilon^m}{\Longrightarrow} \boxed{N_{m+1}^{\mathrm{exit}}.}$$

“Covering-Exchange”

Validation theorem [M.]

1. Assume the existence of

$$\mathcal{F}_\epsilon^\rho \xRightarrow{\varphi_\epsilon(T_\rho,\cdot)} M_\epsilon^{0,0} \xRightarrow{P_\epsilon^{0,0}} M_\epsilon^{0,1} \xRightarrow{P_\epsilon^{0,1}} \cdots \xRightarrow{P_\epsilon^{0,m_0-1}} M_\epsilon^{0,m_0}$$
$$\xRightarrow{P_\epsilon^{0,m_0}} \mathcal{F}_\epsilon^0 \xRightarrow{\varphi_\epsilon(T_0,\cdot)} M_\epsilon^{1,0} \xRightarrow{P_\epsilon^{1,0}} \cdots\cdots \xRightarrow{P_\epsilon^{\rho,m_\rho-1}} M_\epsilon^{\rho,m_\rho} \xRightarrow{P_\epsilon^{\rho,m_\rho}} \mathcal{F}_\epsilon^\rho$$

a loop of Covering-Exchange sequence , $\forall \epsilon \in [0, \epsilon_0]$.
$\to$ the existence of periodic orbits , $\forall \epsilon \in (0, \epsilon_0]$.

2. Assume the existence of

$$\left\{B(S_{\epsilon,u}) \xLeftarrow{\varphi_\epsilon(t,\cdot)} B(S_{\epsilon,u})\right\}_k \xRightarrow{r_1 \circ b_{u,\epsilon}^*} \mathcal{F}_\epsilon^0 \xRightarrow{\varphi_\epsilon(T_0,\cdot)} M_\epsilon^{0,0} \xRightarrow{P_\epsilon^{0,0}} M_\epsilon^{0,1} \xRightarrow{P_\epsilon^{0,1}} \cdots \xRightarrow{P_\epsilon^{0,m_0-1}} M_\epsilon^{0,m_0}$$
$$\xRightarrow{P_\epsilon^{0,m_0}} \mathcal{F}_\epsilon^1 \xRightarrow{\varphi_\epsilon(T_1,\cdot)} M_\epsilon^{1,0} \xRightarrow{P_\epsilon^{1,0}} \cdots\cdots \xRightarrow{P_\epsilon^{\rho,m_\rho-1}} M_\epsilon^{\rho,m_\rho}$$

Covering-Exchange sequence with additional cov. rel. , $\forall \epsilon \in [0, \epsilon_0]$.
$\to$ the existence of heteroclinic orbits, $\forall \epsilon \in (0, \epsilon_0]$.

3. Validation Examples

Example : FitzHugh-Nagumo equation

$$\begin{cases} \dot{u} = v \\ \dot{v} = \delta^{-1}(cv - f(u) + w) \\ \dot{w} = \epsilon c^{-1}(u - \gamma w) \end{cases} \quad f(u) = u(u-a)(1-u) \qquad \textbf{(FN)}$$

Computation Procedures :

1. Slow manifolds via fast-saddle-type blocks and cones
   (Error estimates and bounds of eigenvalues)
2. Slow shadowing and Covering-Exchange : jump
   (Error estimates and bounds of eigenvalues)
3. Covering-Exchange : drop
   (Solving ODEs in the fast time scale)

- m-cones
- invariant foliations
- block on slow mfds

Computation Library :

CAPD (http://capd.ii.uj.edu.pl) : C++ library ver. 3.0

CPU : 1.6GHz Intel Core i5 (Macbook Air 2011), Memory : 4GB 1333 MHz DDR3

Example : FitzHugh-Nagumo equation

$$\begin{cases} \dot{u} = v \\ \dot{v} = \delta^{-1}(cv - f(u) + w) \\ \dot{w} = \epsilon c^{-1}(u - \gamma w) \end{cases} \qquad f(u) = u(u-a)(1-u)$$

$$a = 0.01,\ \gamma = 0,\ \delta = 5.0$$

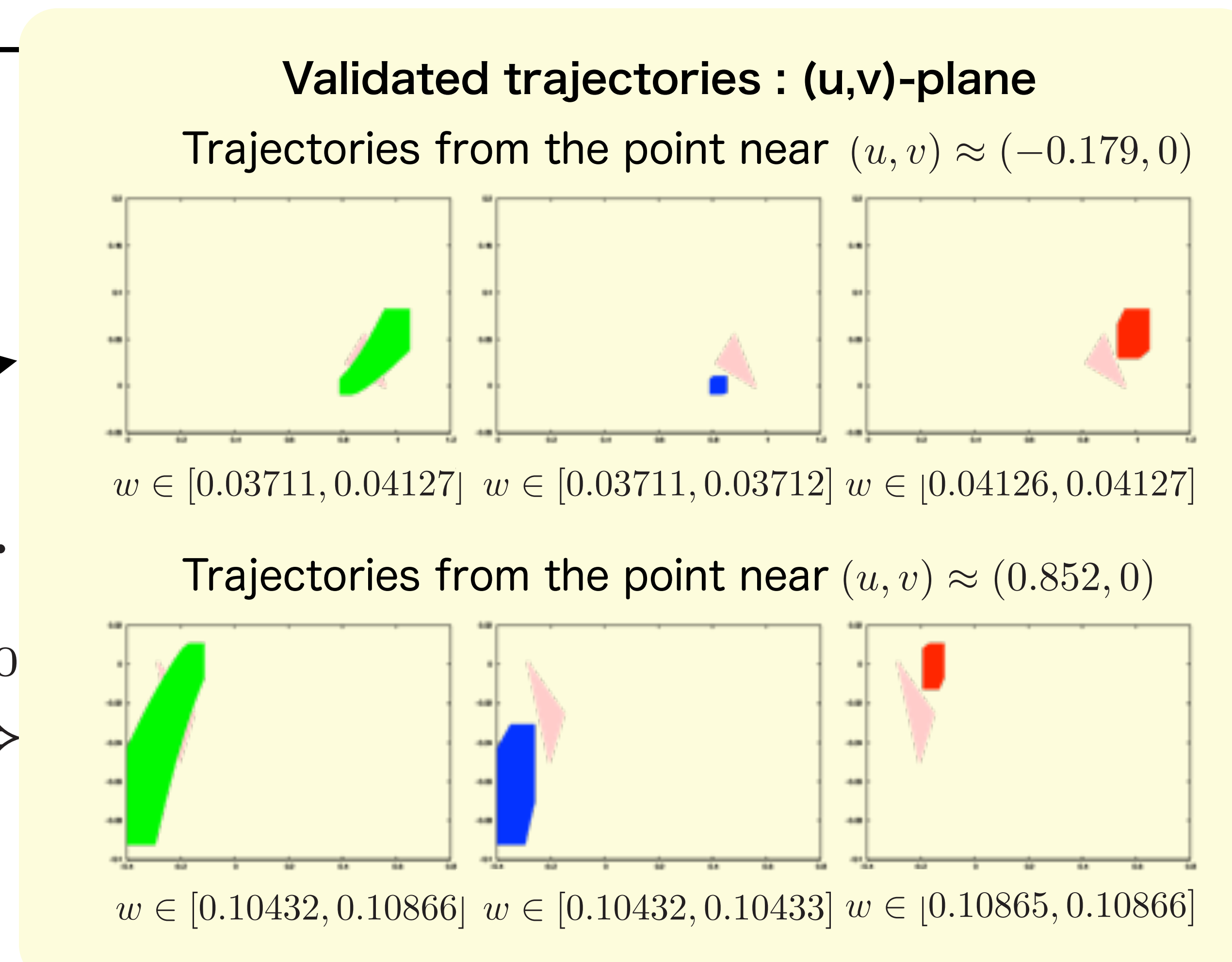
Computer Assisted Result 1 [M. arXiv 1507.01462]

For all $c \in [0.495, 0.505]$ and $\epsilon \in (0, 1.0 \times 10^{-5}]$, there exists a periodic orbit for (FN).

Validation of Covering-Exchange : drop

$$\mathcal{F}_\epsilon^\rho \overset{\varphi_\epsilon(T_\rho,\cdot)}{\Longrightarrow} M_\epsilon^{0,0} \overset{P_\epsilon^{0,0}}{\Longrightarrow} M_\epsilon^{0,1} \overset{P_\epsilon^{0,1}}{\Longrightarrow} \cdots$$

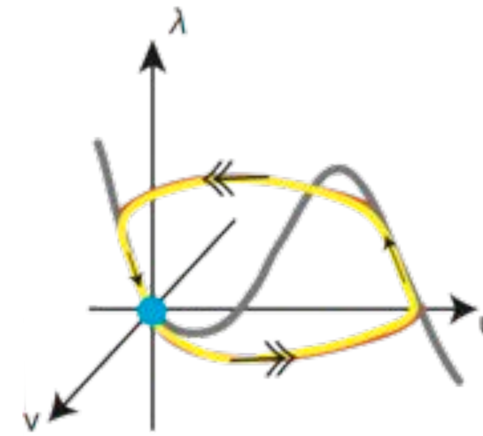
$$\overset{P_\epsilon^{0,m_0}}{\Longrightarrow} \mathcal{F}_\epsilon^0 \overset{\varphi_\epsilon(T_0,\cdot)}{\Longrightarrow} M_\epsilon^{1,0} \overset{P_\epsilon^{1,0}}{\Longrightarrow}$$



Example : FitzHugh-Nagumo equation

$$\begin{cases} \dot{u} = v \\ \dot{v} = \delta^{-1}(cv - f(u) + w) \\ \dot{w} = \epsilon c^{-1}(u - \gamma w) \end{cases} \qquad f(u) = u(u-a)(1-u)$$

$$a = 0.2, \gamma = 7.9, \ \delta = 5.0$$



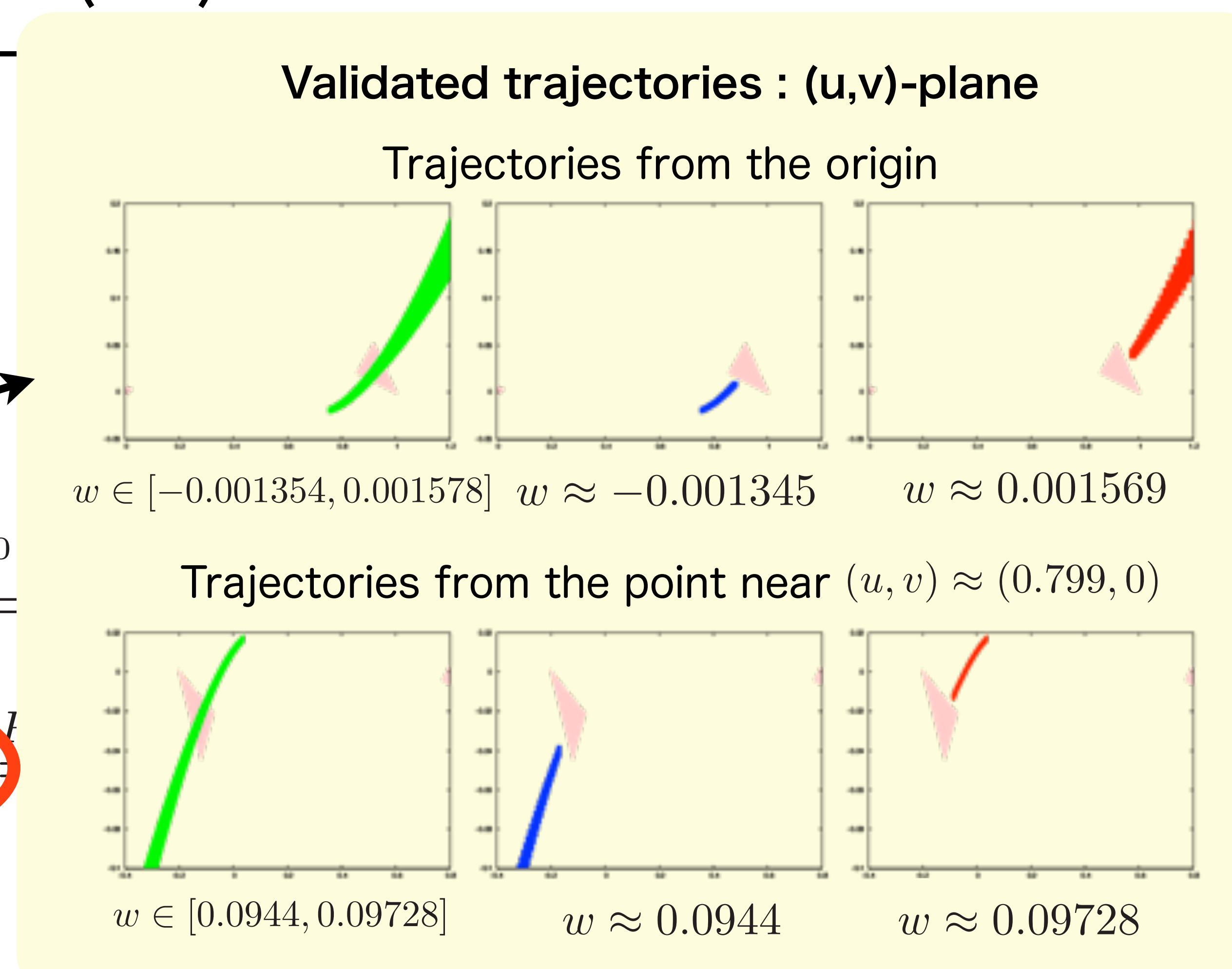
Computer Assisted Result 2 [M. arXiv 1507.01462]

For all $c \in [0.947, 0.948]$ and $\epsilon \in (0, 2.5 \times 10^{-6}]$, there exists a homoclinic orbit of the origin for (FN).

Validation of Covering-Exchange : drop

$$\overset{(t,\cdot)}{=} B(S_{\epsilon,u})\Big\}_k \overset{r_1 \circ b^*_{u,\epsilon}}{\Longrightarrow} \mathcal{F}^0_\epsilon \overset{\varphi_\epsilon(T_0,\cdot)}{\Longrightarrow} M^{0,0}_\epsilon \overset{P^{0,0}_\epsilon}{\Longrightarrow} M^{0,1}_\epsilon \overset{P^0_\epsilon}{\Longrightarrow}$$

$$\overset{P^{0,m_0}_\epsilon}{\Longrightarrow} \mathcal{F}^1_\epsilon \overset{\varphi_\epsilon(T_1,\cdot)}{\Longrightarrow} M^{1,0}_\epsilon$$



Conclusion

“See not objects themselves,
but their neighborhoods.”

Kaname Matsue

Rigorous numerics for fast-slow systems

Topological shadowing approach

Reference

Rigorous numerics for fast-slow systems with one-dimensional slow variable: topological shadowing approach

Kaname Matsue

arXiv 1507.01462