Knots in $\mathbb{R}P^3$ and their lifts

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Motivation

K: a knot in $\mathbb{R}P^3$ with $[K] \neq 0 \in H_1(\mathbb{R}P^3; \mathbb{Z}) \cong \mathbb{Z}/2$.
$\widetilde{K}$: the lift (preimage) of K by $p\colon S^3 \to \mathbb{R}P^3$. $\rightsquigarrow$ $\widetilde{K}$ is a knot in S^3.

Problem (Matveev '15 in ILDT, cf. Fox '61)

Do there exist non-equivalent knots in $\mathbb{R}P^3$ such that their lifts to S^3 are equivalent knots?

Remark

Since $\mathrm{Diff}^+(\mathbb{R}P^3)/\text{diffeotopy} = \{\mathrm{id}_{\mathbb{R}P^3}\}$,
K_0 is ambient isotopic to K_1 iff $(\mathbb{R}P^3, K_0) \cong (\mathbb{R}P^3, K_1)$.

Theorem (Sakuma '86, Boileau-Flapan '87)

Free involutions on (S^3, K) are conjugate to each other.

It follows that the answer to Matveev's question is NO.

Definition

$n(L) := |\{L' \subset \mathbb{R}P^3 \mid \widetilde{L'} \sim L\}| \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ for a link L in S^3.

Problem

- $n(K) = \begin{cases} 0 & \text{if ??,} \\ 1 & \text{if ??,} \end{cases}$ *for a knot* K.
- $\exists ? L$*: a link s.t.* $n(L) > 1$.
- *How about extending it to the lens spaces* $L(p, q)$*?*

Previous studies

Theorem (Hartley '81)

- *$n(T_{p,q}) \geq 1$ iff $2 \nmid pq$, where $T_{p,q}$ is the (p, q)-torus knot.*
- *$n(K) \geq 1$ & $c(K) \leq 10$ iff $K = 0_1$, 10_{124}, 10_{155} or 10_{157}.*

Theorem (Manfredi '14)

"$n_{L(p,q)}(0_1)$" > 1, where $p > 3$ is an odd and $q = (p \pm 1)/2$.

Main results

Definition

G: a group. $G^2 := \langle \{g^2 \mid g \in G\} \rangle < G$.

Theorem (N.)

If $\widetilde{K}$ is a knot, then $\pi_1(S^3 \setminus \widetilde{K}) \cong \pi_1(\mathbb{R}P^3 \setminus K)^2$.

Corollary (cf. Hartley's theorem)

Let K be the (left-/right-handed) trefoil or a knot with $\mathrm{Out}(G(K)) = 1$. Then $n(K) = 0$.

Remark (see Kawauchi (ed.) '96, Kodama-Sakuma '92)

$\mathrm{Out}(G(K)) = 1$ for 9_{32}, 9_{33} or 10_n ($n = 80$, 82–87, 90–95, 102, 106, 107, 110, 117, 119, 148–151, 153) (or their mirror images).

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Subgroup generated by the squares

Lemma

$G^2 := \langle \{g^2 \mid g \in G\} \rangle$ has the following properties:

- $[G, G] \lhd G^2 \lhd G$.
- *If G is finitely generated, then $G/G^2 \cong (\mathbb{Z}/2)^{\oplus n}$ for some $n \geq 0$.*

Example

- $G = (\mathbb{Z}/2)^{\oplus n} \rightsquigarrow G/G^2 \cong (\mathbb{Z}/2)^{\oplus n}$.
- $G = \mathbb{R}_{>0} \rightsquigarrow G/G^2 = 0$.
- $G = \mathbb{Q}_{>0} \rightsquigarrow G/G^2 \cong \bigoplus_{p \colon \text{prime}} \mathbb{Z}/2$.

Let $p\colon S^3 \to \mathbb{R}P^3$ be a double covering.
Let $\widetilde{G} := \pi_1(S^3 \setminus \widetilde{K})$, $G := \pi_1(\mathbb{R}P^3 \setminus K)$.

Proof of $\widetilde{G} \cong G^2$.

$G \twoheadrightarrow G/G^2 \cong (\mathbb{Z}/2)^{\oplus n}$ induces $G^{\mathrm{ab}} \cong \mathbb{Z} \twoheadrightarrow (\mathbb{Z}/2)^{\oplus n}$. $\rightsquigarrow$ $n \leq 1$.
On the other hand, p induces

$$1 \to \widetilde{G} \xrightarrow{p_*} G \to \mathbb{Z}/2 \to 0 \quad \text{(exact)}.$$

Using $G^2 < p_*(\widetilde{G}) < G$, we have

$$[G : G^2] = [G : p_*(\widetilde{G})][p_*(\widetilde{G}) : G^2] \geq 2.$$

$\rightsquigarrow$ $n \geq 1$.
It follows that $[p_*(\widetilde{G}) : G^2] = 1$. $\square$

Definition

G: a group.

- G is *complete* if $\mathrm{Out}(G) = 1$ & $Z(G) = 1$, that is, $G \to \mathrm{Aut}(G)$, $g \mapsto \mathrm{Ad}_g$ is an isomorphism.
- G is an *S-group* if $G \cong {G'}^2$ for some group G'.

Example

- $\mathfrak{S}_n$ is complete unless $n \neq 2, 6$.
- $\mathfrak{A}_n$ is an S-group. Indeed, $\mathfrak{A}_n = (\mathfrak{A}_n)^2 = (\mathfrak{S}_n)^2$.

Our strategy to prove $n(K) = 0$ is to show that $G(K)$ is NOT an S-group. We divide knots into two classes:

- $\mathrm{Out}(G(K)) = 1 \rightsquigarrow G(K)$ is complete & $G(K)^2 \neq G(K)$.
- $K = 3_1 \rightsquigarrow G(K)$ is not complete. (However, $\mathfrak{S}_3$ is complete.)

Proof of $n(K) = 0$ when $\mathrm{Out}(G(K)) = 1$

Lemma (Haugh-MacHale '97)

If G is complete & $G^2 \neq G$, then G is not an S-group.

Fact

- $\mathrm{Out}(G(T_{p,q})) \cong \mathbb{Z}/2$. (Schreier '24)
- $Z(G(K)) = 1 \Leftrightarrow K$ *is neither a torus knot nor the unknot.* (Burde-Zieschang '66)

$\rightsquigarrow$ $G(K)$ *is complete for a knot K with* $\mathrm{Out}(G(K)) = 1$.

Lemma

Let K be a knot in S^3. Then $G(K)^2 \neq G(K)$.

Proof.

Let $G := G(K)$. We have the following commutative diagram.

$$
\begin{tikzcd}
(G^2)^{\mathrm{ab}} \arrow[r, "\mathrm{incl}^{\mathrm{ab}}"] \arrow[dr, dashed] & G^{\mathrm{ab}} \arrow[r, "\cong"] & \mathbb{Z} \\
& (G^{\mathrm{ab}})^2 \arrow[u, hook] \arrow[r, "\cong"] & 2\mathbb{Z} \arrow[u, hook]
\end{tikzcd}
$$

Hence, $(G^{\mathrm{ab}})^2 \hookrightarrow G^{\mathrm{ab}}$ is not surjective, and thus $(G^2)^{\mathrm{ab}} \to G^{\mathrm{ab}}$ is not surjective. Therefore, $G^2 \neq G$. $\square$

Proof of $n(3_1) = 0$

Remark

$G(3_1)$ is not complete since $Z(G(3_1)) \neq 1$.

Definition

G: a group, H: a subgroup of G.
H is *characteristic* if $f(H) < H$ for $\forall f \in \mathrm{Aut}(G)$.

Lemma (Sun '79)

If G: an S-group, $H \lhd G$: characteristic, then G/H is an S-group.

Lemma (Recall)

If G: an S-group, $f\colon G \twoheadrightarrow G'$: a homomorphism, $\operatorname{Ker} f$*: characteristic, then G' is an S-group.*

$G(3_1) \cong \langle a, b \mid a^3 = b^2 \rangle$.
Define $f\colon G(3_1) \twoheadrightarrow \mathfrak{S}_3$ by $a \mapsto (123) = \sigma$, $b \mapsto (12) = \tau$.

Proof of $n(3_1) = 0$.

Assume that $G(3_1)$ is an S-group. Since $\operatorname{Ker} f$ is characteristic (see the next slide), by the above lemma, $\mathfrak{S}_3$ is an S-group. This is a contradiction (see Example in p. 10). $\square$

Lemma

$\mathrm{Ker}(f\colon G(3_1) \twoheadrightarrow \mathfrak{S}_3)$ *is characteristic.*

Proof.

Schreier ('24) proved that $\mathrm{Aut}(G(3_1))$ is generated by $I = \mathrm{Ad}_a$, $J = \mathrm{Ad}_b$ and K, where $K(a) = a^{-1}$, $K(b) = b^{-1}$.
Hence, it suffices to prove $K(\mathrm{Ker}\, f) < \mathrm{Ker}\, f$, namely,
"$g \in \mathrm{Ker}\, f \Rightarrow f(K(g)) = 1$". □

Recall $G(3_1) \cong \langle a, b \mid a^3 = b^2 \rangle$.

Future research

I would like to

- detect when $\pi_1(S^3 \setminus K)$ is not an S-group.
- find a gap between "S-group" and "$n(K) = 1$".
- find a link L with $n(L) > 1$.
- consider $G^p = \langle \{g^p \mid g \in G\} \rangle \lhd G$ in the case of $L(p, q)$.