Formality of the Goldman–Turaev Lie bialgebra and its applications (1), (2)

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Feb 2020, Strasbourg

https://edu.tsuda.ac.jp/~kunotti/slides/Strasbourg2020Feb.pdf

Contents

Based on

- (with N. Kawazumi)
  Handbook of Teichmüller theory, Vol. 5, Chap. 4
- (with A. Alekseev, N. Kawazumi, F. Naef)
  - Adv. Math. (2018) (genus 0, GT formality and KV)
  - 1804.09566 (Higher genus Kashiwara–Vergne theory)
  - 1812.01159 (Goldman bracket and symplectic expansions)

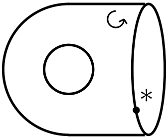
Rough plan:

1. Overview and Goldman bracket
2. Formality question 1 and expansions
3. Turaev cobracket
4. Formality question 2 and expansions

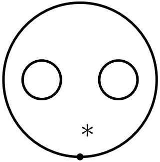
Overview

$\Sigma = \Sigma_{g,n+1}$: surface of genus g with $n+1$ boundary components

$\Sigma_{g,1}$ $(g = 1)$



$\Sigma_{0,n+1}$ $(n = 2)$



$\mathbb{K}$: field of characteristic 0

$$\mathfrak{g}(\Sigma) := \mathbb{K}(\pi_1(\Sigma)/\mathrm{conj}) = \mathbb{K}(\{\text{free loops in } \Sigma\}/\mathrm{homotopy})$$

Loop operations:

- Goldman bracket $[\cdot,\cdot]\colon \mathfrak{g}(\Sigma)^{\otimes 2} \to \mathfrak{g}(\Sigma)$
- (framed) Turaev cobracket $\delta^f\colon \mathfrak{g}(\Sigma) \to \mathfrak{g}(\Sigma)^{\otimes 2}$

Theorem (Goldman, Turaev, Chas)

$(\mathfrak{g}(\Sigma), [\cdot,\cdot], \delta^f)$ is an involutive Lie bialgebra.

$(\mathfrak{g}(\Sigma), [\cdot,\cdot], \delta^f)$: the Goldman–Turaev Lie bialgebra

Fact: One can define the graded version $\mathrm{gr}\,\mathfrak{g}(\Sigma)$.

$\mathrm{gr}\,\mathfrak{g}(\Sigma) =$ necklace Lie (bi)algebra; symplectic/special derivations

Formality question 1

Is $\mathfrak{g}(\Sigma)$ isomorphic to $\mathrm{gr}\,\mathfrak{g}(\Sigma)$ as Lie algebras (after completion)?

Answer: Yes. (2010$\sim$ Kawazumi–K, Massuyeau–Turaev)

Formality question 2

Is $\mathfrak{g}(\Sigma)$ isomorphic to $\mathrm{gr}\,\mathfrak{g}(\Sigma)$ as Lie bialgebras?

Answer: Yes. (Massuyeau ($g = 0$), AKKN, Alekseev–Naef, Hain)

Strategy: refine the formality of free groups by the topology of Σ

A motivation: study of the Johnson homomorphisms

Section 1

Goldman bracket

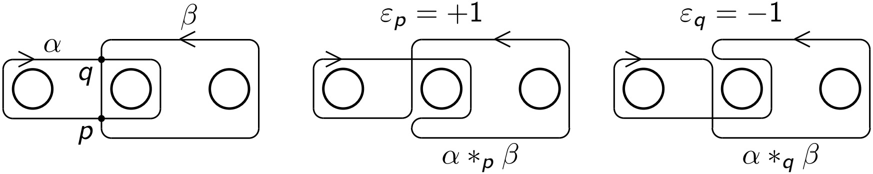
Goldman bracket

Σ: oriented surface

$$\mathfrak{g}(\Sigma) := \mathbb{K}(\pi_1(\Sigma)/\mathrm{conj}) = \mathbb{K}(\{\text{free loops in } \Sigma\}/\mathrm{homotopy})$$

α, β: loops in Σ, in general position

$$[\alpha, \beta] := \sum_{p \in \alpha \cap \beta} \varepsilon_p \, \alpha *_p \beta \quad \in \mathfrak{g}(\Sigma)$$



Theorem (Goldman 1986)

$[\cdot, \cdot]$ is a Lie bracket on $\mathfrak{g}(\Sigma)$.

Background: Wolpert's formula, moduli of flat connections on Σ

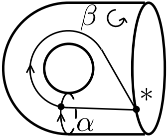
Goldman bracket as derivations

$\pi := \pi_1(\Sigma, *)$, where $* \in \partial\Sigma$

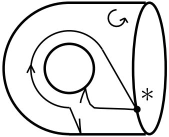
α: free loop, β: based loop, in general position

Definition (Kawazumi–K.)

$$\sigma(\alpha)\beta := \sum_{p \in \alpha\cap\beta} \varepsilon_p \, \beta_{*p}\alpha_p\beta_{p*} \quad \in \mathbb{K}\pi$$



$\sigma(\alpha)\beta =$



Proposition

$\sigma(\alpha)$ is a derivation on the group algebra $\mathbb{K}\pi$ and

$$\sigma : \mathfrak{g}(\Sigma) \to \mathrm{Der}_\partial(\mathbb{K}\pi), \quad \alpha \mapsto \sigma(\alpha)$$

is a Lie homomorphism. (∂ means $\sigma(\alpha)(\partial\Sigma) = 0$.)

Remark: when $n > 0$, replace π with the fundamental groupoid

The graded version of the Goldman bracket

Proposition

There is a graded version of the Lie algebra $(\mathfrak{g}(\Sigma), [\cdot,\cdot])$.

For simplicity, assume $\Sigma = \Sigma_{g,1}$ or $\Sigma = \Sigma_{0,n+1}$.

Filtration of $\mathfrak{g}(\Sigma)$:

The natural projection $\mathbb{K}\pi \to \mathfrak{g}(\Sigma)$ induces

$$\mathfrak{g}(\Sigma) \cong |\mathbb{K}\pi| := \frac{\mathbb{K}\pi}{[\mathbb{K}\pi, \mathbb{K}\pi]} = H_0(\mathbb{K}\pi).$$

$\mathbb{K}\pi$ is filtered by the powers of the augmentation ideal:

$$\mathbb{K}\pi = (I\pi)^0 \supset I\pi \supset (I\pi)^2 \supset \cdots$$

By projection, this induces a filtration of $\mathfrak{g}(\Sigma)$.

$H := \pi^{\mathrm{abel}} \otimes_{\mathbb{Z}} \mathbb{K} \cong H_1(\pi; \mathbb{K})$

Fact: Since $\pi = \pi_1(\Sigma)$ is a free group of finite rank, there is a canonical isomorphism of Hopf algebras:

$$\mathrm{gr}\, \mathbb{K}\pi := \bigoplus_m (I\pi)^m / (I\pi)^{m+1} \cong T(H) = \bigoplus_m H^{\otimes m}$$

$$(\gamma_1 - 1) \cdots (\gamma_m - 1) \longmapsto [\gamma_1] \cdots [\gamma_m]$$

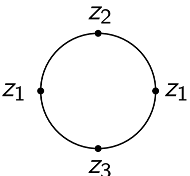
As a $\mathbb{K}$-vector space,

$$\mathrm{gr}\, \mathfrak{g}(\Sigma) \cong |T(H)| := \frac{T(H)}{[T(H), T(H)]} \quad \text{canonically}$$

Fixing basis $\{z_j\}$ for H, $T(H) \cong \mathbb{K}\langle z_1, z_2, \ldots, \rangle$ and

$$|T(H)| \cong \mathrm{Span}_{\mathbb{K}}\{\text{cyclic words in } z_j\}$$

$$|z_1 z_2 z_1 z_3| = |z_2 z_1 z_3 z_1| =$$



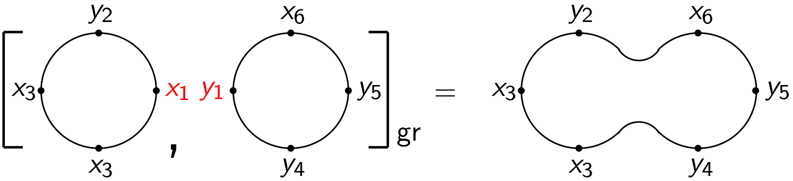
$\mathrm{gr}\,\mathfrak{g}(\Sigma)$ as a Lie algebra: necklace Lie bracket

Suppose that $\Sigma = \Sigma_{g,1}$.

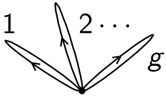
Claim: $[\cdot,\cdot]\colon \mathfrak{g}(\Sigma) \otimes \mathfrak{g}(\Sigma) \to \mathfrak{g}(\Sigma)$ is of degree (-2).

Hence the graded quotient $\mathrm{gr}\,\mathfrak{g}(\Sigma) \cong |T(H)|$ inherits a Lie bracket.

Pictorial formula for $[\cdot,\cdot]_{\mathrm{gr}}$: $(\{x_i, y_i\}_i \subset H$ is a symplectic basis)



Remark: $[\cdot,\cdot]_{\mathrm{gr}}$ is the necklace Lie bracket associated to the quiver



(Bocklandt–Le Bruyn, Ginzburg)

$\mathrm{gr}\,\mathfrak{g}(\Sigma)$ as derivations

Continue the case $\Sigma = \Sigma_{g,1}$.

Recall: the Lie action $\sigma\colon \mathfrak{g}(\Sigma) \to \mathrm{Der}_\partial(\mathbb{K}\pi)$.

Since σ is of degree (-2), $\mathrm{gr}\,\mathfrak{g}(\Sigma)$ acts on $\mathrm{gr}\,\mathbb{K}\pi$ by derivations:

$$\mathrm{gr}\,\sigma\colon \mathrm{gr}\,\mathfrak{g}(\Sigma) \cong |T(H)| \longrightarrow \mathrm{Der}(\mathrm{gr}\,\mathbb{K}\pi) \cong \mathrm{Der}(T(H)).$$

$\langle\cdot,\cdot\rangle\colon H \times H \to \mathbb{K}$: intersection pairing on H

$$\omega := \sum_i (x_i y_i - y_i x_i) \in H^{\otimes 2} \subset T(H) \qquad \text{symplectic form}$$

The Lie algebra of symplectic derivations (Kontsevich, Morita):

$$\mathrm{Der}_\omega(T(H)) := \{D \in \mathrm{Der}(T(H)) \mid D(\omega) = 0\}$$

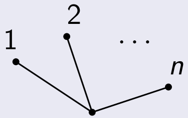
Proposition

$$0 \to \mathbb{K}|1| \to |T(H)| \xrightarrow{\mathrm{gr}\,\sigma} \mathrm{Der}_\omega(T(H)) \to 0 \quad \text{(exact)}.$$

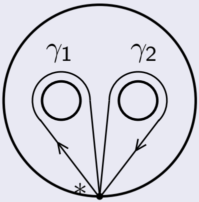
Remarks

If $\Sigma = \Sigma_{0,n+1}$,

1. $[\cdot,\cdot]$ is of degree (-1) (w.r.t. the I-adic filtration)
2. the Lie bracket $[\cdot,\cdot]_{\mathrm{gr}}$ on $\mathrm{gr}\,\mathfrak{g}(\Sigma)$ corresponds to the quiver



3. $\mathrm{gr}\,\mathfrak{g}(\Sigma)$ can be understood as special derivations:



$z_i = [\gamma_i] \in H$

$D \in \mathrm{Der}(T(H))$ is called special if $D(z_i) = [z_i, u_i]$ for any i and $D(z_1 + \cdots + z_n) = 0$.

Section 2

Geometric Johnson homomorphism

Classical Johnson homomorphism

For simplicity, suppose $\Sigma = \Sigma_{g,1}$. (in fact, it is of main interest)

The mapping class group:

$$\mathcal{M} := \{\varphi\colon \Sigma \to \Sigma \mid \text{diffeo}, \varphi|_{\partial\Sigma} = \mathrm{id}|_{\partial\Sigma}\}/\text{isotopy}$$

The Torelli group:

$$\mathcal{I} := \{\varphi \in \mathcal{M} \mid \varphi_* = 1_H\}$$

Dehn–Nielsen theorem

$$\mathrm{DN}\colon \mathcal{M} \xrightarrow{\cong} \mathrm{Aut}_\partial(\pi) = \{\varphi\colon \pi \to \pi \mid \text{auto}, \varphi(\partial\Sigma) = \partial\Sigma\}$$

The action on the l.c.s. of π induces the Johnson filtration

$$\mathcal{M} = \mathcal{M}[0] \supset \mathcal{M}[1] = \mathcal{I} \supset \mathcal{M}[2] \supset \cdots$$

and we obtain the associated graded Lie algebra

$$\mathrm{gr}\,\mathcal{I} := \bigoplus_{k\geq 1} \mathcal{M}[k]/\mathcal{M}[k+1]$$

Johnson homomorphism

$$\tau = \{\tau_k\}_k : \mathrm{gr}\,\mathcal{I} \hookrightarrow \mathfrak{h} = \bigoplus_{k \geq 1} \mathfrak{h}[k] \quad \text{graded Lie homomorphism}$$

where $\mathfrak{h} = \{u \in \mathrm{Der}_\omega(T(H)) \mid \text{positive \& Lie}\} = \mathrm{Der}^+_\omega(L(H))$.
Here, $L(H) \cong \mathrm{FreeLie}(H) \subset T(H)$ is the primitive part of $T(H)$.

Remark. $L(H) = \mathrm{gr}\,\pi \otimes_{\mathbb{Z}} \mathbb{K}$ (w.r.t. the lower central series)

τ is not surjective!

Johnson image problem

Compute $\mathrm{Coker}\,\tau$ (over $\mathbb{Q}$, as Sp-module)

Remark: Hain showed that $\mathrm{Im}\,\tau \otimes \mathbb{Q}$ is generated by degree 1 part.

Morita, Enomoto–Satoh, Conant, Morita–Sakasai–Suzuki,...

Our viewpoint: consider the ungraded version first, use topology, and then take the graded quotient.

Ungraded version of the Johnson homomorphism

$\widehat{\mathbb{K}\pi} := \varprojlim_m \mathbb{K}\pi/(I\pi)^m$ the I-adic completion

$\mathfrak{m}(\pi) := \{u \in \widehat{\mathbb{K}\pi} \mid \Delta(x) = x\hat{\otimes}1 + 1\hat{\otimes}x\}$: the Malcev Lie algebra (Lie bracket: algebra commutator of $\widehat{\mathbb{K}\pi}$)

$\mathcal{M} \curvearrowright \pi$ induces $\mathcal{M} \curvearrowright \widehat{\mathbb{K}\pi}$ and $\mathcal{M} \curvearrowright \mathfrak{m}(\pi)$.

$$\tau^{\mathrm{un}} \colon \mathcal{I} \xrightarrow{\mathsf{DN}} \mathrm{IAut}_\partial(\pi) \hookrightarrow \mathrm{IAut}_\partial^{\mathrm{Hopf}}(\widehat{\mathbb{K}\pi}) \cong \mathrm{IAut}_\partial(\mathfrak{m}(\pi)) \xrightarrow{\log} \mathrm{Der}_\partial^+(\mathfrak{m}(\pi))$$

where "I" means $\mathrm{gr} = \mathrm{id}$ and

$\mathrm{Der}_\partial^+(\mathfrak{m}(\pi)) := \{u \in \mathrm{Der}(\mathfrak{m}(\pi)) \mid \mathrm{gr}\, u = 0, u(\log \partial\Sigma) = 0\}$

τ^{un} is a group emb. (with the BCH product on $\mathrm{Der}_\partial^+(\mathfrak{m}(\pi))$).

Proposition

$\mathrm{gr}\, \tau^{\mathrm{un}} = \tau \colon \mathrm{gr}\, \mathcal{I} \to \mathfrak{h}$

$(\mathrm{gr}\, \mathfrak{m}(\pi) = L(H), [\log \partial\Sigma] \mapsto \omega$ & $\mathrm{gr}\, \mathrm{Der}_\partial^+(\mathfrak{m}(\pi)) = \mathrm{Der}_\omega^+(L(H)) = \mathfrak{h}.)$

Geometric Johnson homomorphism

Recall the Lie action $\sigma\colon \mathfrak{g}(\Sigma) \to \mathrm{Der}_\partial(\mathbb{K}\pi)$.

Theorem (Kawazumi–K.)

$$0 \to \mathbb{K}\mathbf{1} \to \widehat{\mathfrak{g}}(\Sigma) \xrightarrow{\sigma} \mathrm{Der}_\partial(\widehat{\mathbb{K}\pi}) \to 0 \quad (\textit{exact})$$

Remark: the proof uses symplectic expansions.

Definition (geometric Johnson homomorphism)

$$\tau^{\mathrm{geom}}\colon \mathcal{I} \xrightarrow{\tau^{\mathrm{un}}} \mathrm{Der}^+_\partial(\mathfrak{m}(\pi)) \subset \mathrm{Der}_\partial(\widehat{\mathbb{K}\pi}) \xrightarrow{\sigma^{-1}} \frac{\widehat{\mathfrak{g}}(\Sigma)}{\mathbb{K}\mathbf{1}}$$

We obtain an injective group homom

$$\tau^{\mathrm{geom}}\colon \mathcal{I} \longrightarrow L^+(\Sigma)$$

where the target is a pro-nilpotent Lie algebra

$$L^+(\Sigma) := \sigma^{-1}(\mathrm{Der}^+_\partial(\mathfrak{m}(\pi))) \subset \frac{\widehat{\mathfrak{g}}(\Sigma)}{\mathbb{K}\mathbf{1}}$$

Geometric Johnson homomorphism

$$\tau^{\mathrm{geom}} \colon \mathcal{I} \longrightarrow L^+(\Sigma) \subset \frac{\widehat{\mathfrak{g}}(\Sigma)}{\mathbb{K}\mathbf{1}}$$

Example (Dehn twist formula)

$C \subset \Sigma$: separating simple closed curve

$$\tau^{\mathrm{geom}}(t_C) = \frac{1}{2}(\log C)^2 = \left| \left(\sum_{n \geq 1} (-1)^{n-1} \frac{(\gamma - 1)^n}{n} \right)^2 \right| \in L^+(\Sigma),$$

where $\gamma \in \pi_1(\Sigma)$ is freely homotopic to C.

In part (3) and (4), we will show that the Turaev cobracket gives a constraint on the image of τ^{geom}.

Section 3

Formality question 1 and expansions

Review of part (1)

The Goldman bracket

$$[\cdot,\cdot]\colon \mathfrak{g}(\Sigma)^{\otimes 2} \longrightarrow \mathfrak{g}(\Sigma)$$

and its graded version

$$[\cdot,\cdot]_{\mathrm{gr}}\colon |T(H)|^{\otimes 2} \longrightarrow |T(H)|$$

$(\mathrm{gr}\,\mathfrak{g}(\Sigma) = |T(H)|)$.
The Lie subalgebra $L^+(\Sigma) \subset \widehat{\mathfrak{g}}(\Sigma)/\mathbb{K}\mathbf{1}$ serves as the target of the geometric Johnson homomorphism

$$\tau^{\mathrm{geom}}\colon \mathcal{I} \longrightarrow L^+(\Sigma)$$

Formality question 1

Is $\widehat{\mathfrak{g}}(\Sigma)$ isomorphic to $\widehat{\mathrm{gr}}\,\widehat{\mathfrak{g}}(\Sigma) \cong \widehat{\mathrm{gr}}\,\mathfrak{g}(\Sigma)$ as Lie algebras?

Notation: $\widehat{\mathrm{gr}}$ means taking degree completion of gr $(\bigoplus \rightsquigarrow \prod)$
Strategy: start with the formality of $\pi = \pi_1(\Sigma)$

Formality of free groups

$\pi = \langle \gamma_1, \dots, \gamma_n \rangle$: free group of finite rank
$H = \pi^{\mathrm{abel}} \otimes_{\mathbb{Z}} \cong H_1(\pi; \mathbb{K})$
$x_i := [\gamma_i] \in H$: the homology class of γ_i

The I-adic filtration of the group algebra $\mathbb{K}\pi$:

$$\mathbb{K}\pi = (I\pi)^0 \supset I\pi \supset (I\pi)^2 \supset \cdots$$

Recall: There is a canonical isomorphism of Hopf algebras:

$$\mathrm{gr}\,\mathbb{K}\pi := \bigoplus_m (I\pi)^m/(I\pi)^{m+1} \cong T(H) = \mathbb{K}\langle x_1, \dots, x_n \rangle$$

$$(\gamma_i - 1) \longmapsto x_i$$

Group-like expansions

$$\operatorname{gr}\mathbb{K}\pi := \bigoplus_m (I\pi)^m/(I\pi)^{m+1} = T(H) \quad \text{canonically, as Hopf algebras}$$

$\widehat{\mathbb{K}\pi} = \varprojlim_m \mathbb{K}\pi/(I\pi)^m$: the I-adic completion of $\mathbb{K}\pi$

$$\widehat{\operatorname{gr}}\,\widehat{\mathbb{K}\pi} \cong \prod_m (I\pi)^m/(I\pi)^{m+1} = \widehat{T}(H) = \mathbb{K}\langle\langle x_1, \ldots, x_n \rangle\rangle$$

Definition (Massuyeau)

A group-like expansion is an isomorphism

$$\theta\colon \widehat{\mathbb{K}\pi} \xrightarrow{\cong} \widehat{T}(H)$$

of complete Hopf algebras such that $\operatorname{gr}\theta = \mathrm{id}$.

Example

$\theta(\gamma_i) = \exp(x_i) = \sum_{n=0}^{\infty}(1/n!){x_i}^n$.

There are many of them!

$\theta(\gamma_i) = \exp(x_i + \text{arbitrary primitive element of degree} \geq 2)$

Symplectic expansions

Let us take into account the topology of the surface.

Suppose $\Sigma = \Sigma_{g,1}$ and $\pi = \pi_1(\Sigma_{g,1})$.

Definition (Massuyeau)

A group-like expansion θ is called symplectic if

$$\theta(\log \partial\Sigma) = \omega \quad \text{(or equivalently, } \theta(\partial\Sigma) = \exp(\omega).)$$

Practically, θ is specified by the values on generators:

$$\begin{cases} \theta(\log \alpha_i) = x_i + (\text{terms of degree} \geq 2) \\ \theta(\log \beta_i) = y_i + (\text{terms of degree} \geq 2) \end{cases}$$

We must have that

$$\mathrm{BCH}(\cdots \theta(\log \alpha_i), \theta(\log \beta_i), -\theta(\log \alpha_i), -\theta(\log \beta_i) \cdots) = \omega.$$

$(\prod_i \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} = \partial\Sigma.)$

Example of a symplectic expansion

Remark: Symplectic expansions do exist (Kawazumi, Massuyeau).

Example (K., combinatorial method)

$g = 1$ with standard generators α_1, β_1
$x_1 = x,\ y_1 = y$

$$\begin{aligned}\log\theta(\alpha_1) \equiv\ & x + \frac{1}{2}[x,y] + \frac{1}{12}[y,[y,x]] - \frac{1}{8}[x,[x,y]] + \frac{1}{24}[x,[x,[x,y]]] \\ & - \frac{1}{720}[y,[y,[y,[y,x]]]] - \frac{1}{288}[x,[x,[x,[x,y]]]] \\ & - \frac{1}{288}[x,[y,[y,[y,x]]]] - \frac{1}{288}[y,[x,[x,[x,y]]]] \\ & + \frac{1}{144}[[x,y],[y,[y,x]]] + \frac{1}{128}[[x,y],[x,[x,y]]]\end{aligned}$$

modulo terms of degree ≥ 6.

Goldman bracket and symplectic expansions

Any group-like expansion $\theta\colon \widehat{\mathbb{K}\pi} \to \widehat{T}(H)$ induces a filtered $\mathbb{K}$-linear isomorphism

$$\theta\colon \widehat{\mathfrak{g}}(\Sigma) = \frac{\widehat{\mathbb{K}\pi}}{[\widehat{\mathbb{K}\pi}, \widehat{\mathbb{K}\pi}]} \xrightarrow{\cong} \widehat{\mathrm{gr}}\, \widehat{\mathfrak{g}}(\Sigma) = |\widehat{T}(H)| = \frac{\widehat{T}(H)}{[\widehat{T}(H), \widehat{T}(H)]}$$

Theorem (Kawazumi–K.)

If θ is symplectic, then θ is a Lie algebra homomorphism.

This answers the formality question 1 affirmatively.

Remark: several approaches available:

- Kawazumi–K: (co)homology of Hopf algebras
- Massuyeau–Turaev: Fox pairings
- Naef: non-commutative Poisson geometry
- Hain: Hodge theory

Massuyeau–Turaev theorem

Homotopy intersection form (Turaev, Papakyriakopoulos)

For $\alpha, \beta \in \pi$, set $\eta(\alpha, \beta) := \sum_{p \in \alpha \cap \beta} \varepsilon_p \, \alpha_{*p} \beta_{p*} \in \mathbb{K}\pi$.

Theorem (Massuyeau–Turaev)

If θ is symplectic, then the following diagram is commutative.

$$\begin{array}{ccc} \mathbb{K}\pi \times \mathbb{K}\pi & \xrightarrow{\quad \eta \quad} & \mathbb{K}\pi \\ {\scriptstyle \theta \otimes \theta}\downarrow & & \downarrow{\scriptstyle \theta} \\ \widehat{T}(H) \widehat{\otimes} \widehat{T}(H) & \xrightarrow{(\ \overset{\bullet}{\rightsquigarrow}\ ) + \rho_s} & \widehat{T}(H). \end{array}$$

Here, $a_1 \cdots a_m \overset{\bullet}{\rightsquigarrow} b_1 \cdots b_n = \langle a_m \cdot b_1 \rangle a_1 \cdots a_{m-1} b_2 \cdots b_n$ and $\rho_s(a, b) = (a - \varepsilon(a)) s(\omega)(b - \varepsilon(b))$, where $s(\omega) = \frac{1}{\omega} + \frac{1}{(e^{-\omega} - 1)} = -\frac{1}{2} - \frac{\omega}{12} + \frac{\omega^3}{720} - \frac{\omega^5}{30240} + \cdots$. (Bernoulli numbers appear!)

Characterization of the symplectic condition

We have seen that

$$\theta \text{ symplectic} \Rightarrow \theta\colon \widehat{\mathfrak{g}}(\Sigma) \longrightarrow \widehat{\mathrm{gr}}\,\widehat{\mathfrak{g}}(\Sigma) \text{ is Lie}$$

Theorem (AKKN)

Let θ be a group-like expansion. If $\theta\colon \widehat{\mathfrak{g}}(\Sigma) \to \widehat{\mathrm{gr}}\,\widehat{\mathfrak{g}}(\Sigma)$ is a Lie homomorphism, then θ is conjugate to a symplectic expansion: there is a group-like element $g \in \widehat{T}(H)$ such that

$$\theta(\log \partial\Sigma) = g\omega g^{-1}.$$

Remarks:

- it is easier to prove the converse of the MT theorem

$$\theta \text{ symplectic} \Rightarrow \text{ nice description of } \eta$$

- Difficulty lies in characterization of $\exp(\omega)$ in $|\widehat{T}(H)|$.

Proof of "Goldman formality $\Rightarrow$ almost symplectic"

Theorem (Crawley-Boevey–Etingof–Ginzburg, AKKN)

The center of the Lie algebra $(|\widehat{T}(H)|, [\cdot,\cdot]_{\mathrm{gr}})$ *is* $\bigoplus_m \mathbb{K}|\omega^m|$.

Conjugation Theorem (AKKN)

Let $\psi \in \widehat{L}(H)$ (primitive in $\widehat{T}(H)$). If

$$|\exp(\psi)| = |\exp(\omega)| \in |\widehat{T}(H)| = \widehat{T}(H)/[\widehat{T}(H), \widehat{T}(H)],$$

then there is a group-like element $g \in \widehat{T}(H)$ such that

$$\psi = g\omega g^{-1}.$$

Proof of "Goldman formality $\Rightarrow$ almost symplectic":

1. If $\theta\colon \widehat{\mathfrak{g}}(\Sigma) \to |\widehat{T}(H)|$ is Lie, it maps centers to centers.
2. We must have $\theta(\partial\Sigma) = |\exp(\omega)|$.
3. By Conjugation Theorem, $\theta(\log \partial\Sigma) = g\omega g^{-1}$.

More on proof of Conjugation Theorem

Assume that $\psi \in \widehat{L}(H)$ and $|\exp(\psi)| = |\exp(\omega)|$.

Proposition

For any $m \geq 1$, $|\psi^m| = |\omega^m|$.

We want to construct g such that $g\psi g^{-1} = \omega$. We can write

$$\psi = \omega + b_3 + (\text{terms of degree} \geq 4).$$

For any $m \geq 1$, the degree $2m+1$ part of $|\psi^m| = |\omega^m|$ reads:

$$m|b_3\omega^{m-1}| = 0.$$

Key step 1

There is a $u_1 \in H$ such that $b_3 = [\omega, u_1]$.

Then,

$$\psi_1 := e^{u_1}\psi e^{-u_1} \equiv \omega + b_3 + [u_1, \omega] = \omega \quad \text{mod terms of degree} \geq 4$$

Inductive step

Suppose that we have

$$\psi_{n-1} = \omega + b_{n+2} + (\text{terms of degree} \geq n+3)$$

Key step n

Let $b_{n+2} \in \widehat{L}(H)$ be of degree $n+2$ such that $|b_{n+2}\omega^m| =$ for any $m \gg 0$. Then,

$$b_{n+2} = [\omega, u_n]$$

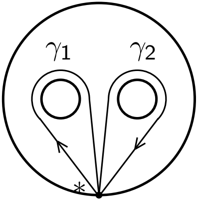
for some element $u_n \in \widehat{L}(H)$ of degree n.

We can constuct a sequence $\psi_0 = \psi, \psi_1, \psi_2, \psi_3, \ldots,$ such that ψ_n is conjugate to ψ_{n-1} and $\{\psi_n\}_n$ converges to ω. Then, for some group-like element g

$$g\psi g^{-1} = \lim_{n\to\infty} \psi_n = \omega.$$

The case of genus 0

Let $\Sigma = \Sigma_{0,n+1}$.



$z_i = [\gamma_i] \in H$

Based on works by Habegger–Masbaum, Enriquez, Alekseev–Enriquez–Torossian, Massuyeau introduced:

Definition

A group-like expansion $\theta\colon \widehat{\mathbb{K}\pi} \to \widehat{T}(H)$ is called special if $\theta(\gamma_i) = g_i \exp(z_i) g_i^{-1}$ for any i and

$$\theta(\log \partial\Sigma) = z_1 + \cdots + z_n.$$

Remark: All the statements for $\Sigma_{g,1}$ ("symplectic $\Rightarrow$ Goldman formality", its converse, and the Massuyeau–Turaev theorem) generalize to $\Sigma_{0,n+1}$, as well as to the case of general $(g, n+1)$.

The space of symplectic/special expansions

$\Theta := \{\text{group-like expansions}\}$

$\exp \mathrm{Der}^+(\widehat{L}(H)) \curvearrowright \Theta$ by post-composition, freely and transitively

The case $\Sigma = \Sigma_{g,1}$
$\Theta_{\mathrm{symp}} := \{\theta \in \Theta \mid \theta \text{ is symplectic}\}$ $(\neq \emptyset)$
$\widehat{\mathfrak{h}} = \mathrm{Der}^+_\omega(\widehat{L}(H))$: completion of $\mathfrak{h} = \bigoplus_k \mathfrak{h}[k]$

$$\exp \widehat{\mathfrak{h}} \curvearrowright \Theta_{\mathrm{symp}}$$

The size of Θ and Θ_{symp} can be read from the degree k parts

$$\mathrm{Der}^+_k(\widehat{L}(H)) \cong H \otimes L_{k+1}(H),$$

$$\mathfrak{h}[k] \cong \mathrm{Ker}(H \otimes L_{k+1}(H) \xrightarrow{[\cdot,\cdot]} L_{k+2}(H))$$

When $\Sigma = \Sigma_{0,n+1}$, the set of special expansions is a torsor under the exponentiation of

$$\mathfrak{sder} := \{u \in \mathrm{Der}^+(\widehat{L}(H)) \mid u(z_i) = [z_i, u_i] \ \& \ u(z_1 + \cdots + z_n) = 0\}$$

the case $\Sigma_{1,1}$:

k	$\mathrm{rk}\,\mathrm{Der}^+_k(\widehat{L}(H))$	$\mathrm{rk}\,\mathfrak{h}[k]$
1	2	0
2	4	1
3	6	0
4	12	3
5	18	0
6	36	6
7	60	4
8	112	13
9	198	12
10	372	37

the case of $\Sigma_{0,3}$:

k	$\mathrm{rk}\,\mathfrak{sder}[k]$
1	1
2	0
3	1
4	0
5	3
6	0
7	6
8	4
9	13
10	12

Remark: $\pi_1(\Sigma_{1,1}) \cong F_2 \cong \pi_1(\Sigma_{0,3})$

Summary of Part (1) and (2)

$(\mathfrak{g}(\Sigma), [\cdot,\cdot], \delta^f)$: the Goldman–Turaev Lie bialgebra

In Part (1) and (2), we focused on the Goldman bracket.

The graded version $(\mathrm{gr}\,\mathfrak{g}(\Sigma), [\cdot,\cdot]_{\mathrm{gr}})$:

$$\mathrm{gr}\,\mathfrak{g}(\Sigma) = |T(H)|, \quad [\cdot,\cdot]_{\mathrm{gr}} = \text{necklace Lie bracket}$$

Symplectic(/special) expansions:

Goldman formality $\leftrightsquigarrow$ symplectic condition $\theta(\partial\Sigma) = \exp(\omega)$

$\exp\widehat{\mathfrak{h}} \curvearrowright \Theta_{\mathrm{symp}} = \{\text{symplectic expansions}\} \subsetneq \{\text{group-like expansions}\}$

Next question:

$$\Theta_{\mathrm{symp}} \supsetneq\ ??$$